# 14. SOME SPECIAL INTEGRALS

## Three Special Integrals

THEOREM (i) $\int \frac{dx}{(a^2 - x^2)} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + C.$

(ii) $\int \frac{dx}{(x^2 - a^2)} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C.$

(iii) $\int \frac{dx}{(x^2 + a^2)} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C.$

PROOF (i) $\int \frac{dx}{(a^2 - x^2)} = \int \frac{dx}{(a+x)(a-x)}$

$$= \int \frac{1}{2a} \cdot \left\{ \frac{(a-x)+(a+x)}{(a+x)(a-x)} \right\} dx = \frac{1}{2a} \cdot \left[ \int \frac{dx}{(a+x)} + \int \frac{dx}{(a-x)} \right]$$

$$= \frac{1}{2a} \cdot [\log |a+x| - \log |a-x|] + C$$

$$= \frac{1}{2a} \cdot \log \left| \frac{a+x}{a-x} \right| + C.$$

$\therefore \int \frac{dx}{(a^2 - x^2)} = \frac{1}{2a} \cdot \log \left| \frac{a+x}{a-x} \right| + C.$

(ii) $\int \frac{dx}{(x^2 - a^2)} = \int \frac{dx}{(x-a)(x+a)}$

$$= \int \frac{1}{2a} \cdot \left\{ \frac{(x+a)-(x-a)}{(x-a)(x+a)} \right\} dx = \frac{1}{2a} \cdot \left[ \int \frac{dx}{(x-a)} - \int \frac{dx}{(x+a)} \right]$$

$$= \frac{1}{2a} \cdot [\log |x-a| - \log |x+a|] + C = \frac{1}{2a} \cdot \log \left| \frac{x-a}{x+a} \right| + C.$$

$\therefore \int \frac{dx}{(x^2 - a^2)} = \frac{1}{2a} \cdot \log \left| \frac{x-a}{x+a} \right| + C.$

(iii) $\int \frac{dx}{(x^2 + a^2)} = \frac{1}{a^2} \cdot \int \frac{dx}{\left(1 + \frac{x^2}{a^2}\right)}$

$$= \frac{1}{a^2} \cdot \int \frac{a\,dt}{(1+t^2)} \quad \left[\text{putting } \frac{x}{a} = t \text{ and } dx = a\,dt\right]$$

$$= \frac{1}{a} \tan^{-1} t + C = \frac{1}{a} \tan^{-1} \frac{x}{a} + C.$$

$$\therefore \quad \int \frac{dx}{(x^2+a^2)} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C.$$

REMARK If we have an integral of the form $\int \frac{dx}{(ax^2+bx+c)}$ then we put the denominator in the form $[(x+\alpha)^2 \pm \beta^2]$ and then integrate.

## SOLVED EXAMPLES

EXAMPLE 1 *Evaluate:*

(i) $\int \frac{dx}{(1-4x^2)}$ (ii) $\int \frac{dx}{(32-2x^2)}$ (iii) $\int \frac{x^2}{(1-x^6)}dx$

SOLUTION We have

(i) $$\int \frac{dx}{(1-4x^2)} = \frac{1}{4}\cdot\int \frac{dx}{\left(\frac{1}{4}-x^2\right)}$$

$$= \frac{1}{4}\cdot\int \frac{dx}{\left\{\left(\frac{1}{2}\right)^2 - x^2\right\}}$$

$$= \frac{1}{4}\cdot\frac{1}{\left(2\times\frac{1}{2}\right)}\cdot\log\left|\frac{\frac{1}{2}+x}{\frac{1}{2}-x}\right| + C$$

$$\left[\because \int \frac{dx}{(a^2-x^2)} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C\right]$$

$$= \frac{1}{4}\log\left|\frac{1+2x}{1-2x}\right| + C.$$

(ii) $$\int \frac{dx}{(32-2x^2)} = \frac{1}{2}\cdot\int \frac{dx}{(16-x^2)}$$

$$= \frac{1}{2}\cdot\int \frac{dx}{\{(4)^2-x^2\}}$$

$$= \frac{1}{2}\cdot\frac{1}{(2\times 4)}\log\left|\frac{4+x}{4-x}\right| + C$$

$$\left[\because \int \frac{dx}{(a^2-x^2)} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C\right]$$

$$= \frac{1}{16}\log\left|\frac{4+x}{4-x}\right| + C.$$

(iii) Putting $x^3 = t$ and $3x^2dx = dt$, we get

$$\int \frac{x^2}{(1-x^6)}dx = \frac{1}{3}\cdot\int \frac{1}{(1-t^2)}dt$$

$$= \frac{1}{3} \cdot \frac{1}{(2 \times 1)} \cdot \log \left| \frac{1+t}{1-t} \right| + C$$

$$\left[ \because \int \frac{dx}{(a^2 - x^2)} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + C \right]$$

$$= \frac{1}{6} \log \left| \frac{1+x^3}{1-x^3} \right| + C.$$

EXAMPLE 2 *Evaluate:* (i) $\int \frac{\sin x}{(1 - 4\cos^2 x)} dx$ (ii) $\int \frac{\text{cosec}^2 x}{(1 - \cot^2 x)} dx$

SOLUTION (i) Putting $\cos x = t$ and $-\sin x \, dx = dt$, we get

$$\int \frac{\sin x}{(1 - 4\cos^2 x)} dx = -\int \frac{dt}{(1 - 4t^2)}$$

$$= -\frac{1}{4} \cdot \int \frac{dt}{\left(\frac{1}{4} - t^2\right)} = -\frac{1}{4} \cdot \int \frac{dt}{\left\{\left(\frac{1}{2}\right)^2 - t^2\right\}}$$

$$= -\frac{1}{4} \times \frac{1}{\left(2 \times \frac{1}{2}\right)} \log \left| \frac{\frac{1}{2} + t}{\frac{1}{2} - t} \right| + C$$

$$= -\frac{1}{4} \log \left| \frac{1+2t}{1-2t} \right| + C = -\frac{1}{4} \log \left| \frac{1 + 2\cos x}{1 - 2\cos x} \right| + C.$$

(ii) Putting $\cot x = t$ and $-\text{cosec}^2 x \, dx = dt$, we get

$$\int \frac{\text{cosec}^2 x}{(1 - \cot^2 x)} dx = -\int \frac{dt}{(1 - t^2)} = -\frac{1}{(2 \times 1)} \log \left| \frac{1+t}{1-t} \right| + C$$

$$= -\frac{1}{2} \log \left| \frac{1 + \cot x}{1 - \cot x} \right| + C.$$

EXAMPLE 3 *Evaluate:*

(i) $\int \frac{dx}{(9x^2 - 1)}$ (ii) $\int \frac{x}{(x^4 - 9)} dx$ (iii) $\int \frac{x^2}{(x^2 - 9)} dx$

SOLUTION We have

(i) $$\int \frac{dx}{(9x^2 - 1)} = \frac{1}{9} \cdot \int \frac{dx}{\left(x^2 - \frac{1}{9}\right)}$$

$$= \frac{1}{9} \cdot \int \frac{dx}{\left\{x^2 - \left(\frac{1}{3}\right)^2\right\}}$$

$$= \frac{1}{9} \cdot \frac{1}{\left(2 \times \frac{1}{3}\right)} \log \left| \frac{x - \frac{1}{3}}{x + \frac{1}{3}} \right| + C$$

$$\left[ \because \int \frac{dx}{(x^2 - a^2)} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C \right]$$

$$= \frac{1}{6} \log \left| \frac{3x-1}{3x+1} \right| + C.$$

(ii) Putting $x^2 = t$ and $2x\ dx = dt$, we get

$$\int \frac{x}{(x^4 - 9)} dx = \frac{1}{2} \int \frac{dt}{(t^2 - 9)} = \frac{1}{2} \cdot \int \frac{dt}{\{t^2 - (3)^2\}}$$

$$= \frac{1}{2} \cdot \frac{1}{(2 \times 3)} \log \left| \frac{t-3}{t+3} \right| + C$$

$$= \frac{1}{12} \log \left| \frac{x^2 - 3}{x^2 + 3} \right| + C.$$

(iii) $$\int \frac{x^2}{(x^2 - 9)} dx = \int \left\{ 1 + \frac{9}{x^2 - 9} \right\} dx$$

$$= \int dx + 9 \int \frac{dx}{\{x^2 - (3)^2\}}$$

$$= x + 9 \cdot \left[ \frac{1}{(2 \times 3)} \log \left| \frac{x-3}{x+3} \right| \right] + C$$

$$= x + \frac{3}{2} \log \left| \frac{x-3}{x+3} \right| + C.$$

EXAMPLE 4 *Evaluate* $\displaystyle\int \frac{dx}{(4 + 25x^2)}$.

SOLUTION We have

$$\int \frac{dx}{(4 + 25x^2)} = \frac{1}{25} \cdot \int \frac{dx}{\left(\frac{4}{25} + x^2\right)}$$

$$= \frac{1}{25} \cdot \int \frac{dx}{\{(2/5)^2 + x^2\}}$$

$$= \frac{1}{25} \cdot \frac{1}{(2/5)} \tan^{-1} \frac{x}{(2/5)} + C \quad \left[ \because \int \frac{dx}{(a^2 + x^2)} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C \right]$$

$$= \frac{1}{10} \tan^{-1} \left( \frac{5x}{2} \right) + C.$$

EXAMPLE 5 *Evaluate* $\displaystyle\int \frac{3x}{(1 + 2x^4)} dx.$

SOLUTION Putting $x^2 = t$ and $2x\,dx = dt$, we get

$$\int \frac{3x}{(1+2x^4)}dx = \frac{3}{2}\cdot\int \frac{dt}{(1+2t^2)}$$

$$= \frac{3}{4}\cdot\int \frac{dt}{\left(\frac{1}{2}+t^2\right)} = \frac{3}{4}\cdot\int \frac{dt}{\left\{\left(\frac{1}{\sqrt{2}}\right)^2 + t^2\right\}}$$

$$= \frac{3}{4}\cdot\frac{1}{(1/\sqrt{2})}\tan^{-1}\frac{t}{(1/\sqrt{2})} + C$$

$$= \frac{3}{2\sqrt{2}}\tan^{-1}(\sqrt{2}t) + C = \frac{3}{2\sqrt{2}}\tan^{-1}(\sqrt{2}x^2) + C.$$

EXAMPLE 6 *Evaluate* $\int \frac{dx}{(1+5\sin^2 x)}\cdot$

SOLUTION On dividing num. and denom. by $\cos^2 x$, we get

$$\int \frac{dx}{(1+5\sin^2 x)} = \int \frac{(1/\cos^2 x)}{\left(\frac{1}{\cos^2 x}+5\cdot\frac{\sin^2 x}{\cos^2 x}\right)}dx$$

$$= \int \frac{\sec^2 x}{(\sec^2 x + 5\tan^2 x)}dx = \int \frac{\sec^2 x}{\{(1+\tan^2 x)+5\tan^2 x\}}dx$$

$$= \int \frac{\sec^2 x}{(1+6\tan^2 x)}dx = \int \frac{dt}{(1+6t^2)}, \text{ where } \tan x = t$$

$$= \frac{1}{6}\int \frac{dt}{\left(\frac{1}{6}+t^2\right)} = \frac{1}{6}\cdot\int \frac{dt}{\left\{\left(\frac{1}{\sqrt{6}}\right)^2+t^2\right\}}$$

$$= \frac{1}{6}\cdot\frac{1}{(1/\sqrt{6})}\tan^{-1}\frac{t}{(1/\sqrt{6})} + C = \frac{1}{\sqrt{6}}\tan^{-1}(\sqrt{6}t) + C$$

$$= \frac{1}{\sqrt{6}}\tan^{-1}(\sqrt{6}\tan x) + C.$$

EXAMPLE 7 *Evaluate* $\int \frac{dx}{(2+\sin^2 x)}\cdot$

SOLUTION On dividing num. and denom. by $\cos^2 x$, we get

$$\int \frac{dx}{(2+\sin^2 x)} = \int \frac{\sec^2 x}{(2\sec^2 x + \tan^2 x)}dx = \int \frac{\sec^2 x}{\{2(1+\tan^2 x)+\tan^2 x\}}dx$$

$$= \int \frac{\sec^2 x}{2+3\tan^2 x}dx = \int \frac{dt}{(2+3t^2)}, \text{ where } \tan x = t$$

$$= \frac{1}{3}\cdot\int \frac{dt}{\left(t^2+\frac{2}{3}\right)} = \frac{1}{3}\cdot\int \frac{dt}{\left(\sqrt{\frac{2}{3}}\right)^2+t^2}$$

$$= \frac{1}{3} \cdot \frac{1}{\sqrt{\frac{2}{3}}} \cdot \tan^{-1} \frac{t}{\left(\sqrt{\frac{2}{3}}\right)} + C = \frac{1}{\sqrt{6}} \tan^{-1} \frac{\sqrt{3}t}{\sqrt{2}} + C$$

$$= \frac{1}{\sqrt{6}} \tan^{-1}\left(\frac{\sqrt{3}\tan x}{\sqrt{2}}\right) + C.$$

EXAMPLE 8 *Evaluate* $\int \frac{dx}{(x^2 + 6x + 13)}$.

SOLUTION We have

$$\int \frac{dx}{(x^2 + 6x + 13)} = \int \frac{dx}{\{(x^2 + 6x + 9) + 4\}}$$

$$= \int \frac{dx}{\{(x + 3)^2 + 2^2\}} = \int \frac{dt}{(t^2 + 2^2)}, \text{ where } (x + 3) = t$$

$$= \frac{1}{2} \tan^{-1} \frac{t}{2} + C$$

$$= \frac{1}{2} \tan^{-1} \frac{1}{2}(x + 3) + C.$$

EXAMPLE 9 *Evaluate* $\int \frac{dx}{(x^2 + 8x + 20)}$. **[CBSE 2002C]**

SOLUTION We have

$$\int \frac{dx}{(x^2 + 8x + 20)} = \int \frac{dx}{\{(x + 4)^2 + 2^2\}} = \int \frac{dt}{(t^2 + 2^2)}, \text{ where } (x + 4) = t$$

$$= \frac{1}{2} \tan^{-1} \frac{t}{2} + C$$

$$= \frac{1}{2} \tan^{-1} \frac{1}{2}(x + 4) + C.$$

EXAMPLE 10 *Evaluate* $\int \frac{dx}{(9x^2 - 12x + 8)}$.

SOLUTION We have

$$(9x^2 - 12x + 8) = 9\left(x^2 - \frac{4}{3}x + \frac{8}{9}\right)$$

$$= 9\left\{\left(x^2 - \frac{4}{3}x + \frac{4}{9}\right) - \frac{4}{9} + \frac{8}{9}\right\} = 9\left\{\left(x - \frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2\right\}$$

$$\therefore \int \frac{dx}{(9x^2 - 12x + 8)} = \frac{1}{9} \cdot \int \frac{dx}{\left\{\left(x - \frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2\right\}}$$

$$= \frac{1}{9} \cdot \frac{1}{\left(\frac{2}{3}\right)} \tan^{-1}\left\{\frac{\left(x - \frac{2}{3}\right)}{\left(\frac{2}{3}\right)}\right\} + C = \frac{1}{6} \tan^{-1}\left(\frac{3x - 2}{2}\right) + C.$$

EXAMPLE 11 *Evaluate* $\int \frac{x}{(x^4 - x^2 + 1)}\,dx.$ **[CBSE 2003, '07C]**

SOLUTION Putting $x^2 = t$ and $2x\,dx = dt$, we get

$$\int \frac{x}{(x^4 - x^2 + 1)}\,dx = \frac{1}{2}\cdot\int \frac{dt}{(t^2 - t + 1)} = \frac{1}{2}\cdot\int \frac{dt}{\left\{\left(t - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2\right\}}$$

$$= \frac{1}{2}\cdot\frac{1}{\left(\frac{\sqrt{3}}{2}\right)}\tan^{-1}\frac{\left(t - \frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)} + C$$

$$= \frac{1}{\sqrt{3}}\cdot\tan^{-1}\left(\frac{2t - 1}{\sqrt{3}}\right) + C = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2 - 1}{\sqrt{3}}\right) + C.$$

EXAMPLE 12 *Evaluate* $\int \frac{dx}{(2x^2 + x - 1)}$.

SOLUTION We have

$$\int \frac{dx}{(2x^2 + x - 1)} = \frac{1}{2}\cdot\int \frac{dx}{\left(x^2 + \frac{1}{2}x - \frac{1}{2}\right)}$$

$$= \frac{1}{2}\int \frac{dx}{\left[\left\{x^2 + \frac{1}{2}x + \left(\frac{1}{4}\right)^2\right\} - \frac{1}{16} - \frac{1}{2}\right]} = \frac{1}{2}\int \frac{dx}{\left[\left(x + \frac{1}{4}\right)^2 - \left(\frac{3}{4}\right)^2\right]}$$

$$= \frac{1}{2}\cdot\frac{1}{2\cdot\frac{3}{4}}\log\left|\frac{\left(x + \frac{1}{4}\right) - \frac{3}{4}}{\left(x + \frac{1}{4}\right) + \frac{3}{4}}\right| + C = \frac{1}{3}\log\left|\frac{2x - 1}{2(x + 1)}\right| + C.$$

EXAMPLE 13 *Evaluate* $\int \frac{dx}{(3x^2 + 13x - 10)}$.

SOLUTION We have

$$(3x^2 + 13x - 10) = 3\left(x^2 + \frac{13}{3}x - \frac{10}{3}\right)$$

$$= 3\left\{\left(x + \frac{13}{6}\right)^2 - \frac{169}{36} - \frac{10}{3}\right\} = 3\left\{\left(x + \frac{13}{6}\right)^2 - \frac{289}{36}\right\}$$

$$= 3\left\{\left(x + \frac{13}{6}\right)^2 - \left(\frac{17}{6}\right)^2\right\}.$$

$$\therefore\ \int \frac{dx}{(3x^2 + 13x - 10)} = \int \frac{dx}{3\left\{\left(x + \frac{13}{6}\right)^2 - \left(\frac{17}{6}\right)^2\right\}}$$

$$= \frac{1}{3}\int \frac{dt}{\left\{t^2 - \left(\frac{17}{6}\right)^2\right\}}, \text{ where } \left(x + \frac{13}{6}\right) = t$$

$$= \frac{1}{3} \cdot \frac{1}{\left(2 \times \frac{17}{6}\right)} \log \left|\frac{t - \frac{17}{6}}{t + \frac{17}{6}}\right| + C$$

$$\left[\because \int \frac{dx}{(x^2 - a^2)} = \frac{1}{2a} \log \left|\frac{x-a}{x+a}\right|\right]$$

$$= \frac{1}{17} \log \left|\frac{6t - 17}{6t + 17}\right| + C$$

$$= \frac{1}{17} \log \left|\frac{6\left(x + \frac{13}{6}\right) - 17}{6\left(x + \frac{13}{6}\right) + 17}\right| = \frac{1}{17} \log \left|\frac{6x - 4}{6x + 30}\right| + C$$

$$= \frac{1}{17} \log \left|\frac{3x - 2}{3x + 15}\right| + C = \frac{1}{17} \log \left|\frac{(3x - 2)}{3(x + 5)}\right| + C$$

$$= \frac{1}{17}\left\{\log \frac{1}{3} + \log \left|\frac{3x - 2}{x + 5}\right|\right\} + C$$

$$= \frac{1}{17} \log \left|\frac{3x - 2}{x + 5}\right| + k,$$

where $\frac{1}{17} \log \frac{1}{3} + C = k = \text{constant}.$

EXAMPLE 14 *Evaluate* $\int \frac{dx}{(1 + x - x^2)}$.

SOLUTION We have

$$\int \frac{dx}{(1 + x - x^2)} = -\int \frac{dx}{(x^2 - x - 1)}$$

$$= -\int \frac{dx}{\left\{\left(x^2 - x + \frac{1}{4}\right) - \frac{5}{4}\right\}} = -\int \frac{dx}{\left\{\left(x - \frac{1}{2}\right)^2 - \left(\frac{\sqrt{5}}{2}\right)^2\right\}}$$

$$= \int \frac{dx}{\left\{\left(\frac{\sqrt{5}}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2\right\}} = \int \frac{dx}{\left\{\left(\frac{\sqrt{5}}{2}\right)^2 - u^2\right\}}, \text{ where } \left(x - \frac{1}{2}\right) = u$$

$$= \frac{1}{\left(2 \times \frac{\sqrt{5}}{2}\right)} \cdot \log \left|\frac{\frac{\sqrt{5}}{2} + u}{\frac{\sqrt{5}}{2} - u}\right| + C$$

$$= \frac{1}{\sqrt{5}} \log \left| \frac{\sqrt{5} + 2u}{\sqrt{5} - 2u} \right| + C = \frac{1}{\sqrt{5}} \log \left| \frac{\sqrt{5} + 2\left(x - \frac{1}{2}\right)}{\sqrt{5} - 2\left(x - \frac{1}{2}\right)} \right| + C$$

$$= \frac{1}{\sqrt{5}} \log \left| \frac{\sqrt{5} + 2x - 1}{\sqrt{5} - 2x + 1} \right| + C = \frac{1}{\sqrt{5}} \log \left| \frac{(\sqrt{5} - 1) + 2x}{(\sqrt{5} + 1) - 2x} \right| + C.$$

EXAMPLE 15 *Evaluate* $\int \frac{dx}{(5 - 8x - x^2)}$.

SOLUTION We have

$$\int \frac{dx}{(5 - 8x - x^2)} = -\int \frac{dx}{(x^2 + 8x - 5)}$$

$$= -\int \frac{dx}{\{(x^2 + 8x + 16) - 21\}} = -\int \frac{dx}{\{(x + 4)^2 - (\sqrt{21})^2\}}$$

$$= \int \frac{dx}{\{(\sqrt{21})^2 - (x + 4)^2\}} = \int \frac{dt}{\{(\sqrt{21})^2 - t^2\}},$$

where $(x + 4) = t$

$$= \frac{1}{2\sqrt{21}} \cdot \log \left| \frac{\sqrt{21} + t}{\sqrt{21} - t} \right| + C$$

$$= \frac{1}{2\sqrt{21}} \cdot \log \left| \frac{\sqrt{21} + 4 + x}{\sqrt{21} - 4 - x} \right| + C.$$

EXAMPLE 16 *Evaluate* $\int \frac{dx}{(1 - 6x - 9x^2)}$.

SOLUTION We have

$$\int \frac{dx}{(1 - 6x - 9x^2)} = -\int \frac{dx}{(9x^2 + 6x - 1)} = -\frac{1}{9}\int \frac{dx}{\left(x^2 + \frac{2}{3}x - \frac{1}{9}\right)}$$

$$= -\frac{1}{9} \cdot \int \frac{dx}{\left\{\left(x^2 + \frac{2}{3}x + \frac{1}{9}\right) - \frac{2}{9}\right\}}$$

$$= -\frac{1}{9} \cdot \int \frac{dx}{\left\{\left(x + \frac{1}{3}\right)^2 - \left(\frac{\sqrt{2}}{3}\right)^2\right\}} = \frac{1}{9} \cdot \int \frac{dx}{\left\{\left(\frac{\sqrt{2}}{3}\right)^2 - \left(x + \frac{1}{3}\right)^2\right\}}$$

$$= \frac{1}{9} \cdot \int \frac{dx}{\left\{\left(\frac{\sqrt{2}}{3}\right)^2 - t^2\right\}}, \text{ where } \left(x + \frac{1}{3}\right) = t$$

$$= \frac{1}{9} \cdot \frac{1}{2 \cdot \frac{\sqrt{2}}{3}} \log \left| \frac{\frac{\sqrt{2}}{3} + t}{\frac{\sqrt{2}}{3} - t} \right| + C = \frac{1}{6\sqrt{2}} \log \left| \frac{\sqrt{2} + 3t}{\sqrt{2} - 3t} \right| + C$$

$$= \frac{1}{6\sqrt{2}} \log \left| \frac{\sqrt{2} + 3\left(x + \frac{1}{3}\right)}{\sqrt{2} - 3\left(x + \frac{1}{3}\right)} \right| + C$$

$$= \frac{1}{6\sqrt{2}} \log \left| \frac{\sqrt{2} + 1 + 3x}{\sqrt{2} - 1 - 3x} \right| + C.$$

EXAMPLE 17 *Evaluate* $\int \frac{\cos x}{(\sin^2 x + 4\sin x + 5)} dx.$

SOLUTION Putting $\sin x = t$ and $\cos x \, dx = dt$, we get

$$\int \frac{\cos x}{(\sin^2 x + 4\sin x + 5)} dx = \int \frac{dt}{(t^2 + 4t + 5)} = \int \frac{dt}{\{(t^2 + 4t + 4) + 1\}}$$

$$= \int \frac{dt}{\{(t+2)^2 + 1^2\}} = \int \frac{du}{(u^2 + 1)},$$

where $u = (t + 2)$

$$= \tan^{-1} u + C = \tan^{-1}(t + 2) + C$$

$$= \tan^{-1}(\sin x + 2) + C.$$

EXAMPLE 18 *Evaluate* $\int \frac{e^x}{(e^{2x} + 6e^x + 5)} dx.$

SOLUTION Putting $e^x = t$ and $e^x dx = dt$, we get

$$\int \frac{e^x}{e^{2x} + 6e^x + 5} dx = \int \frac{dt}{(t^2 + 6t + 5)} = \int \frac{dt}{\{(t^2 + 6t + 9) - 4\}}$$

$$= \int \frac{dt}{\{(t+3)^2 - 2^2\}} = \int \frac{du}{(u^2 - 2^2)}, \text{ where } (t + 3) = u$$

$$= \frac{1}{(2 \times 2)} \log \left| \frac{u-2}{u+2} \right| + C = \frac{1}{4} \log \left| \frac{t+3-2}{t+3+2} \right| + C$$

$$= \frac{1}{4} \log \left| \frac{t+1}{t+5} \right| + C = \frac{1}{4} \log \left| \frac{e^x + 1}{e^x + 5} \right| + C.$$

**Integrals of the form** $\int \frac{(px + q)}{(ax^2 + bx + c)} dx.$

**METHOD** Let $(px + q) = A \cdot \frac{d}{dx}(ax^2 + bx + c) + B$

Find $A$ and $B$.

Now, the integrand so obtained can be integrated easily.

EXAMPLE 19 *Evaluate* $\int \frac{(5x - 2)}{(1 + 2x + 3x^2)} dx.$ **[CBSE 2013]**

SOLUTION Let $(5x - 2) = A \cdot \frac{d}{dx}(1 + 2x + 3x^2) + B.$

Then, $(5x-2) = A(6x+2) + B$. ... (i)

Comparing the coefficients of like powers of $x$ on both sides, we get

$6A = 5$ and $2A + B = -2$.

This gives $A = \frac{5}{6}$ and $B = \frac{-11}{3}$.

$$\therefore \quad I = \int \frac{\left\{\frac{5}{6}(6x+2) - \frac{11}{3}\right\}}{(1+2x+3x^2)} dx$$

$$= \frac{5}{6} \cdot \int \frac{6x+2}{(1+2x+3x^2)} dx - \frac{11}{3} \int \frac{dx}{(3x^2+2x+1)}$$

$$= \frac{5}{6} \log |1+2x+3x^2| - \frac{11}{3} \cdot \frac{1}{3} \int \frac{dx}{\left(x^2 + \frac{2}{3}x + \frac{1}{3}\right)}$$

$$= \frac{5}{6} \log |1+2x+3x^2| - \frac{11}{9} \cdot \int \frac{dx}{\left\{\left(x+\frac{1}{3}\right)^2 + \left(\frac{1}{3} - \frac{1}{9}\right)\right\}}$$

$$= \frac{5}{6} \log |1+2x+3x^2| - \frac{11}{9} \cdot \int \frac{dx}{\left\{\left(x+\frac{1}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2\right\}} + C$$

$$= \frac{5}{6} \log |1+2x+3x^2| - \frac{11}{9} \cdot \frac{1}{\left(\frac{\sqrt{2}}{3}\right)} \tan^{-1} \left\{\frac{x+\frac{1}{3}}{\left(\frac{\sqrt{2}}{3}\right)}\right\} + C$$

$$= \frac{5}{6} \log |1+2x+3x^2| - \frac{11}{3\sqrt{2}} \tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C.$$

EXAMPLE 20 *Evaluate* $\int \frac{(3x+1)}{(2x^2-2x+3)} dx.$ **[CBSE 2006]**

SOLUTION Let $(3x+1) = A \cdot \frac{d}{dx}(2x^2 - 2x + 3) + B$. Then,

$(3x+1) = A(4x-2) + B$ ... (i)

Comparing the coefficients of like powers of $x$, we get

$(4A = 3 \text{ and } B - 2A = 1) \Rightarrow \left(A = \frac{3}{4} \text{ and } B = \frac{5}{2}\right)$.

$$\therefore \quad \int \frac{(3x+1)}{(2x^2-2x+3)} dx = \int \frac{A \cdot (4x-2) + B}{(2x^2-2x+3)}$$

$$= \int \frac{\frac{3}{4} \cdot (4x-2) + \frac{5}{2}}{(2x^2-2x+3)} dx = \frac{3}{4} \cdot \int \frac{(4x-2)}{(2x^2-2x+3)} dx + \frac{5}{2} \int \frac{dx}{2\left(x^2 - x + \frac{3}{2}\right)}$$

$$= \frac{3}{4}\log|2x^2 - 2x + 3| + \frac{5}{4}\cdot\int \frac{dx}{\left\{\left(x^2 - x + \frac{1}{4}\right) + \frac{5}{4}\right\}}$$

$$= \frac{3}{4}\log|2x^2 - 2x + 3| + \frac{5}{4}\cdot\int \frac{dx}{\left\{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{5}}{2}\right)^2\right\}}$$

$$= \frac{3}{4}\log|2x^2 - 2x + 3| + \frac{5}{4}\cdot\frac{1}{\left(\frac{\sqrt{5}}{2}\right)}\tan^{-1}\left\{\frac{\left(x - \frac{1}{2}\right)}{\left(\frac{\sqrt{5}}{2}\right)}\right\} + C$$

$$= \frac{3}{4}\log|2x^2 - 2x + 3| + \frac{\sqrt{5}}{2}\tan^{-1}\left(\frac{2x - 1}{\sqrt{5}}\right) + C.$$

EXAMPLE 21 *Evaluate* $\int \frac{(2x+1)}{(4 - 3x - x^2)}dx.$ **[CBSE 2004C]**

SOLUTION Let $(2x + 1) = A\cdot\frac{d}{dx}(4 - 3x - x^2) + B.$

Then, $(2x + 1) = A(-3 - 2x) + B$ … (i)

Comparing the coefficients of like terms, we get

$$(-2A = 2 \text{ and } -3A + B = 1) \Rightarrow (A = -1, B = -2).$$

$$\therefore \int \frac{(2x+1)}{(4 - 3x - x^2)}dx = \int \left\{\frac{(-1)\cdot(-3 - 2x) - 2}{(4 - 3x - x^2)}\right\}dx$$

$$= -\int \frac{(-3 - 2x)}{(4 - 3x - x^2)}dx - 2\int \frac{dx}{(4 - 3x - x^2)}$$

$$= -\log|4 - 3x - x^2| + 2\int \frac{dx}{(x^2 + 3x - 4)}$$

$$= -\log|4 - 3x - x^2| + 2\int \frac{dx}{\left(x + \frac{3}{2}\right)^2 - \left(4 + \frac{9}{4}\right)}$$

$$= -\log|4 - 3x - x^2| + 2\int \frac{dx}{\left\{\left(x + \frac{3}{2}\right)^2 - \left(\frac{5}{2}\right)^2\right\}}$$

$$= -\log|4 - 3x - x^2| + \frac{2}{\left(2\times\frac{5}{2}\right)}\log\left|\frac{\left(x + \frac{3}{2}\right) - \frac{5}{2}}{\left(x + \frac{3}{2}\right) + \frac{5}{2}}\right| + C$$

$$= -\log|4 - 3x - x^2| + \frac{2}{5}\log\left|\frac{x - 1}{x + 4}\right| + C.$$

EXAMPLE 22 *Evaluate* $\int \left( \frac{x^2 + 5x + 3}{x^2 + 3x + 2} \right) dx.$

SOLUTION We have

$$\frac{(x^2 + 5x + 3)}{(x^2 + 3x + 2)} = \left\{ 1 + \frac{(2x+1)}{x^2 + 3x + 2} \right\}$$

$$\Rightarrow \int \frac{(x^2 + 5x + 3)}{(x^2 + 3x + 2)} dx = \int dx + \int \frac{(2x+1)}{(x^2 + 3x + 2)} dx. \qquad \ldots \text{(i)}$$

Let $(2x+1) = A \cdot \frac{d}{dx}(x^2 + 3x + 2) + B$. Then,

$$(2x+1) = A(2x+3) + B. \qquad \ldots \text{(ii)}$$

Comparing the coefficients of like powers of $x$, we get

$$(2A = 2 \text{ and } 3A + B = 1) \Rightarrow (A = 1 \text{ and } B = -2).$$

$\therefore$ $(2x+1) = (2x+3) - 2.$

$$\therefore \; I = x + \int \frac{(2x+1)}{(x^2 + 3x + 2)} dx = x + \int \frac{\{(2x+3) - 2\}}{(x^2 + 3x + 2)} dx$$

$$= x + \int \frac{(2x+3)}{(x^2 + 3x + 2)} dx - 2\int \frac{dx}{(x^2 + 3x + 2)}$$

$$= x + \log|x^2 + 3x + 2| - 2\int \frac{dx}{\left\{ \left( x^2 + 3x + \frac{9}{4} \right) + \left( 2 - \frac{9}{4} \right) \right\}}$$

$$= x + \log|x^2 + 3x + 2| - 2\int \frac{dx}{\left\{ \left( x + \frac{3}{2} \right)^2 - \left( \frac{1}{2} \right)^2 \right\}}$$

$$= x + \log|x^2 + 3x + 2| - 2 \cdot \frac{1}{\left( 2 \times \frac{1}{2} \right)} \log \left| \frac{x + \frac{3}{2} - \frac{1}{2}}{x + \frac{3}{2} + \frac{1}{2}} \right| + C$$

$$= x + \log|x^2 + 3x + 2| - 2\log \left| \frac{x+1}{x+2} \right| + C.$$

**Integrals of the form** $\int \frac{dx}{a + b\cos^2 x}, \int \frac{dx}{a + b\sin^2 x}$

**and** $\int \frac{dx}{a\cos^2 x + b\sin x\cos x + c\sin^2 x}.$

**METHOD** In each such an integral, we divide the numerator and denominator by $\cos^2 x$ and put $\tan x = t$, $\sec^2 x \, dx = dt$ and then integrate.

EXAMPLE 23 *Evaluate* $\int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x}.$ **[CBSE 2003C]**

SOLUTION Dividing the numerator and the denominator of the given integrand by $\cos^2 x$, we get

$$\int \frac{dx}{(a^2 \sin^2 x + b^2 \cos^2 x)} = \int \frac{\sec^2 x}{a^2 \tan^2 x + b^2}\, dx$$

$$= \int \frac{dt}{(a^2 t^2 + b^2)} \qquad [\text{putting } \tan x = t]$$

$$= \frac{1}{a^2} \int \frac{dt}{\left[t^2 + \left(\frac{b}{a}\right)^2\right]} = \frac{1}{a^2} \cdot \frac{1}{\left(\frac{b}{a}\right)} \tan^{-1} \frac{t}{\left(\frac{b}{a}\right)} + C$$

$$= \frac{1}{ab} \tan^{-1}\left(\frac{at}{b}\right) + C = \frac{1}{ab} \tan^{-1}\left(\frac{a \tan x}{b}\right) + C.$$

EXAMPLE 24 *Evaluate:*

(i) $\int \frac{dx}{(1 + 3\sin^2 x)}$ (ii) $\int \frac{dx}{(3 + 2\cos^2 x)}$

SOLUTION (i) Dividing the numerator and denominator by $\cos^2 x$, we get

$$\int \frac{dx}{(1 + 3\sin^2 x)} = \int \frac{\sec^2 x}{\sec^2 x + 3\tan^2 x}\, dx = \int \frac{\sec^2 x}{(1 + 4\tan^2 x)}\, dx$$

$$= \int \frac{dt}{(1 + 4t^2)} \qquad [\text{putting } \tan x = t]$$

$$= \frac{1}{4} \int \frac{dt}{\left[t^2 + \left(\frac{1}{2}\right)^2\right]} = \frac{1}{4} \cdot \frac{1}{(1/2)} \tan^{-1} \frac{t}{(1/2)} + C$$

$$= \frac{1}{2} \tan^{-1}(2t) + C = \frac{1}{2} \tan^{-1}(2\tan x) + C.$$

(ii) Dividing the numerator and denominator by $\cos^2 x$, we get

$$\int \frac{dx}{(3 + 2\cos^2 x)} = \int \frac{\sec^2 x}{(3\sec^2 x + 2)}\, dx = \int \frac{\sec^2 x}{5 + 3\tan^2 x}\, dx$$

$$= \int \frac{dt}{(5 + 3t^2)} \qquad [\text{putting } \tan x = t]$$

$$= \frac{1}{3} \int \frac{dt}{\left[t^2 + \left(\frac{5}{3}\right)\right]} = \frac{1}{3} \cdot \int \frac{dt}{\left[t^2 + \left(\frac{\sqrt{5}}{\sqrt{3}}\right)^2\right]}$$

$$= \frac{1}{3} \cdot \frac{1}{\left(\frac{\sqrt{5}}{\sqrt{3}}\right)} \tan^{-1} \frac{t}{\left(\frac{\sqrt{5}}{\sqrt{3}}\right)} + C = \frac{1}{\sqrt{15}} \tan^{-1}\left(\frac{\sqrt{3}t}{\sqrt{5}}\right) + C$$

$$= \frac{1}{\sqrt{15}} \tan^{-1}\left(\frac{\sqrt{3}\tan x}{\sqrt{5}}\right) + C.$$

EXAMPLE 25 *Evaluate* $\int \frac{dx}{(4\sin^2 x + 5\cos^2 x)}$.

SOLUTION On dividing the numerator and denominator by $\cos^2 x$, we get

$$\int \frac{dx}{(4\sin^2 x + 5\cos^2 x)} = \int \frac{\sec^2 x}{(4\tan^2 x + 5)} dx$$

$$= \int \frac{dt}{4t^2 + 5} \quad \text{[putting } \tan x = t\text{]}$$

$$= \frac{1}{4}\int \frac{dt}{\left(t^2 + \frac{5}{4}\right)} = \frac{1}{4}\cdot\int \frac{dt}{\left[t^2 + \left(\frac{\sqrt{5}}{2}\right)^2\right]}$$

$$= \frac{1}{4}\cdot\frac{1}{\left(\frac{\sqrt{5}}{2}\right)}\tan^{-1}\frac{t}{\left(\frac{\sqrt{5}}{2}\right)} + C$$

$$= \frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{2t}{\sqrt{5}}\right) + C = \frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{2\tan x}{\sqrt{5}}\right) + C.$$

EXAMPLE 26 *Evaluate* $\int \frac{dx}{(1 + 3\sin^2 x + 8\cos^2 x)}$.

SOLUTION On dividing the numerator and denominator by $\cos^2 x$, we get

$$\int \frac{dx}{(1 + 3\sin^2 x + 8\cos^2 x)} = \int \frac{\sec^2 x\, dx}{\sec^2 x + 3\tan^2 x + 8}$$

$$= \int \frac{\sec^2 x}{9 + 4\tan^2 x} dx = \int \frac{dt}{9 + 4t^2}$$

[putting $\tan x = t$]

$$= \frac{1}{4}\int \frac{dt}{\left(t^2 + \frac{9}{4}\right)} = \frac{1}{4}\cdot\int \frac{dt}{\left[t^2 + \left(\frac{3}{2}\right)^2\right]}$$

$$= \frac{1}{4}\cdot\frac{1}{(3/2)}\cdot\tan^{-1}\left\{\frac{t}{(3/2)}\right\} + C$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{2t}{3}\right) + C = \frac{1}{6}\tan^{-1}\left(\frac{2\tan x}{3}\right) + C.$$

EXAMPLE 27 *Evaluate* $\int \frac{\sin x}{\sin 3x} dx$.

SOLUTION We have

$$\int \frac{\sin x}{\sin 3x} dx = \int \frac{\sin x}{(3\sin x - 4\sin^3 x)} dx$$

$$= \int \frac{1}{(3 - 4\sin^2 x)} dx \quad \text{[dividing num. and denom. by } \sin x\text{]}$$

$$= \int \frac{\sec^2 x}{3\sec^2 x - 4\tan^2 x}\,dx \quad \text{[dividing num. and denom. by } \cos^2 x]$$

$$= \int \frac{\sec^2 x}{3(1+\tan^2 x) - 4\tan^2 x}\,dx = \int \frac{\sec^2 x}{(3-\tan^2 x)}\,dx$$

$$= \int \frac{dt}{(3-t^2)}, \text{ where } \tan x = t \text{ and } \sec^2 x\,dx = dt$$

$$= \int \frac{dt}{[(\sqrt{3})^2 - t^2]} = \frac{1}{2\sqrt{3}} \log \left| \frac{\sqrt{3}+t}{\sqrt{3}-t} \right| + C$$

$$= \frac{1}{2\sqrt{3}} \log \left| \frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x} \right| + C.$$

EXAMPLE 28 *Evaluate* $\int \frac{\cos x}{\cos 3x}\,dx.$

SOLUTION We have

$$\int \frac{\cos x}{\cos 3x}\,dx = \int \frac{\cos x}{(4\cos^3 x - 3\cos x)}\,dx$$

$$= \int \frac{dx}{4\cos^2 x - 3} = \int \frac{dx}{4\cos^2 x - 3(\sin^2 x + \cos^2 x)}$$

$$= \int \frac{dx}{\cos^2 x - 3\sin^2 x} = \int \frac{\sec^2 x}{1 - 3\tan^2 x}\,dx$$

[on dividing the num. and denom. by $\cos^2 x$]

$$= \int \frac{dt}{1-3t^2}, \text{ where } \tan x = t \text{ and } \sec^2 x\,dx = dt$$

$$= \frac{1}{3} \cdot \int \frac{dt}{\left(\frac{1}{3} - t^2\right)} = \frac{1}{3} \cdot \int \frac{dt}{\left[\left(\frac{1}{\sqrt{3}}\right)^2 - t^2\right]} = \frac{1}{3} \cdot \frac{1}{2 \cdot \frac{1}{\sqrt{3}}} \log \left| \frac{\frac{1}{\sqrt{3}} + t}{\frac{1}{\sqrt{3}} - t} \right| + C$$

$$= \frac{1}{2\sqrt{3}} \log \left| \frac{1+\sqrt{3}\,t}{1-\sqrt{3}\,t} \right| + C = \frac{1}{2\sqrt{3}} \log \left| \frac{1+\sqrt{3}\tan x}{1-\sqrt{3}\tan x} \right| + C.$$

EXAMPLE 29 *Evaluate* $\int \frac{dx}{(2+\cos x)}.$

SOLUTION We have

$$\int \frac{dx}{(2+\cos x)} = \int \frac{dx}{1+(1+\cos x)} = \int \frac{dx}{1+2\cos^2(x/2)} = \int \frac{\sec^2(x/2)\,dx}{\sec^2(x/2)+2}$$

[dividing the num. and denom. by $\cos^2(x/2)$]

$$= \int \frac{\sec^2(x/2)}{3 + \tan^2(x/2)} dx = 2\int \frac{dt}{3 + t^2}, \text{ where } \tan(x/2) = t$$

$$= 2 \cdot \int \frac{dt}{(\sqrt{3})^2 + t^2} = 2 \cdot \frac{1}{\sqrt{3}} \tan^{-1} \frac{t}{\sqrt{3}} + C$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left[\frac{\tan(x/2)}{\sqrt{3}}\right] + C.$$

## Some More Special Integrals

EXAMPLE 30 *Evaluate* $\int \frac{(x^2+1)}{(x^4+1)} dx.$ **[CBSE 2006C, '07, '11C]**

SOLUTION We have

$$\int \left(\frac{x^2+1}{x^4+1}\right) dx = \int \frac{\left(1 + \frac{1}{x^2}\right)}{\left(x^2 + \frac{1}{x^2}\right)} dx \quad \text{[dividing num. and denom. by } x^2\text{]}$$

$$= \int \frac{\left(1 + \frac{1}{x^2}\right)}{\left(x - \frac{1}{x}\right)^2 + 2} dx$$

$$= \int \frac{dt}{[t^2 + (\sqrt{2})^2]}, \text{ where } \left(x - \frac{1}{x}\right) = t \text{ and } \left(1 + \frac{1}{x^2}\right) dx = dt$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + C = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x - \frac{1}{x}}{\sqrt{2}}\right) + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) + C.$$

EXAMPLE 31 *Evaluate* $\int \frac{(x^2+4)}{(x^4+16)} dx.$ **[CBSE 2007C, '11C]**

SOLUTION We have

$$\int \left(\frac{x^2+4}{x^4+16}\right) dx = \int \frac{\left(1 + \frac{4}{x^2}\right)}{\left(x^2 + \frac{16}{x^2}\right)} dx \quad \text{[dividing num. and denom. by } x^2\text{]}$$

$$= \int \frac{\left(1 + \frac{4}{x^2}\right)}{\left(x - \frac{4}{x}\right)^2 + 8} dx$$

$$= \int \frac{dt}{(t^2+8)} \quad \left[\text{putting } \left(x - \frac{4}{x}\right) = t \text{ and } \left(1 + \frac{4}{x^2}\right) dx = dt\right]$$

$$= \int \frac{dt}{t^2 + (\sqrt{8})^2} = \frac{1}{\sqrt{8}} \tan^{-1}\left(\frac{t}{\sqrt{8}}\right) + C$$

$$= \frac{1}{\sqrt{8}} \tan^{-1} \frac{\left(x - \frac{4}{x}\right)}{\sqrt{8}} + C = \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{x^2 - 4}{2\sqrt{2}x}\right) + C.$$

EXAMPLE 32 *Evaluate* $\int \frac{(x^2-1)}{(x^4+x^2+1)} dx.$

SOLUTION We have

$$\int \frac{(x^2-1)}{(x^4+x^2+1)} dx = \int \frac{\left(1 - \frac{1}{x^2}\right)}{\left(x^2 + \frac{1}{x^2} + 1\right)} dx$$

[dividing num. and denom. by $x^2$]

$$= \int \frac{\left(1 - \frac{1}{x^2}\right)}{\left[\left(x + \frac{1}{x}\right)^2 - 1\right]} dx = \int \frac{dt}{(t^2-1)}$$

$\left[\text{putting } \left(x + \frac{1}{x}\right) = t \text{ and } \left(1 - \frac{1}{x^2}\right) dx = dt\right]$

$$= \frac{1}{2} \log \left|\frac{t-1}{t+1}\right| + C = \frac{1}{2} \log \left|\frac{x + \frac{1}{x} - 1}{x + \frac{1}{x} + 1}\right| + C$$

$$= \frac{1}{2} \log \left|\frac{x^2 - x + 1}{x^2 + x + 1}\right| + C.$$

EXAMPLE 33 *Evaluate* $\int \frac{dx}{(x^4+1)}.$

SOLUTION We have

$$\int \frac{dx}{(x^4+1)} = \int \frac{(x^2+1) - (x^2-1)}{2(x^4+1)} dx$$

$$= \frac{1}{2} \int \frac{(x^2+1)}{(x^4+1)} dx - \frac{1}{2} \int \frac{(x^2-1)}{(x^4+1)} dx$$

$$= \frac{1}{2}\left[\int \frac{\left(1+\frac{1}{x^2}\right)}{\left(x^2+\frac{1}{x^2}\right)} dx - \int \frac{\left(1-\frac{1}{x^2}\right)}{\left(x^2+\frac{1}{x^2}\right)} dx\right]$$

[dividing num. and denom. of each integral by $x^2$]

$$= \frac{1}{2}\left[\int \frac{\left(1+\frac{1}{x^2}\right)}{\left[\left(x-\frac{1}{x}\right)^2+2\right]} dx - \int \frac{\left(1-\frac{1}{x^2}\right)}{\left[\left(x+\frac{1}{x}\right)^2-2\right]} dx\right]$$

$$= \frac{1}{2}\left[\int \frac{dt}{[t^2+(\sqrt{2})^2]} - \int \frac{du}{[u^2-(\sqrt{2})^2]}\right]$$

[putting $\left(x-\frac{1}{x}\right)=t$ in the 1st integral, and $\left(x+\frac{1}{x}\right)=u$ in the 2nd]

$$= \frac{1}{2}\left\{\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right) - \frac{1}{2\sqrt{2}}\log\left|\frac{u-\sqrt{2}}{u+\sqrt{2}}\right|\right\} + C$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right) - \frac{1}{4\sqrt{2}}\log\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right| + C$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) - \frac{1}{4\sqrt{2}}\log\left|\frac{x^2+1-\sqrt{2}x}{x^2+1+\sqrt{2}x}\right| + C.$$

EXAMPLE 34 *Evaluate* $\int \frac{x^2}{(x^4+x^2+1)} dx.$ **[CBSE 2008C]**

SOLUTION We have

$$I = \frac{1}{2}\int \frac{2x^2}{(x^4+x^2+1)} dx$$

$$= \frac{1}{2}\cdot\int \frac{(x^2-1)+(x^2+1)}{(x^4+x^2+1)} dx$$

$$= \frac{1}{2}\int \frac{(x^2-1)}{(x^4+x^2+1)} dx + \frac{1}{2}\int \frac{(x^2+1)}{(x^4+x^2+1)} dx$$

$$= \frac{1}{2}\int \frac{\left(1-\frac{1}{x^2}\right)}{\left(x^2+1+\frac{1}{x^2}\right)} dx + \frac{1}{2}\int \frac{\left(1+\frac{1}{x^2}\right)}{\left(x^2+1+\frac{1}{x^2}\right)} dx$$

[on dividing num. and denom. of each by $x^2$]

$$= \frac{1}{2}\int \frac{\left(1-\frac{1}{x^2}\right)}{\left\{\left(x+\frac{1}{x}\right)^2-1\right\}}dx + \frac{1}{2}\int \frac{\left(1+\frac{1}{x^2}\right)}{\left\{\left(x-\frac{1}{x}\right)^2+(\sqrt{3})^2\right\}}dx$$

$$= \frac{1}{2}\int \frac{du}{(u^2-1)} + \frac{1}{2}\int \frac{dv}{\{v^2+(\sqrt{3})^2\}}$$

$$\left\{\begin{array}{l}\text{putting } \left(x+\frac{1}{x}\right)=u \text{ and } \left(1-\frac{1}{x^2}\right)=du \text{ in } I_1, \\ \quad\text{and } \left(x-\frac{1}{x}\right)=v \text{ and } \left(1+\frac{1}{x^2}\right)dx=dv \text{ in } I_2\end{array}\right\}$$

$$= \frac{1}{2}\log\left|\frac{u-1}{u+1}\right| + \frac{1}{2}\cdot\frac{1}{\sqrt{3}}\tan^{-1}\frac{v}{\sqrt{3}} + C$$

$$= \frac{1}{2}\log\left|\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1}\right| + \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{3}x}\right) + C$$

$$= \frac{1}{2}\log\left|\frac{x^2-x+1}{x^2+x+1}\right| + \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x^2+1}{\sqrt{3}x}\right) + C.$$

EXAMPLE 35 *Evaluate* $\int\sqrt{\cot x}\,dx.$

SOLUTION Put $\cot x = t^2$ so that $-\text{cosec}^2 x\,dx = 2t\,dt$ or $dx = \frac{-2t}{(1+t^4)}dt.$

$$\therefore \int\sqrt{\cot x}\,dx = -\int\frac{2t^2}{(t^4+1)}dt$$

$$= -\int\frac{[(t^2+1)+(t^2-1)]}{(t^4+1)}dt = -\int\frac{(t^2+1)}{(t^4+1)}dt - \int\frac{(t^2-1)}{(t^4+1)}dt$$

$$= -\int\frac{\left(1+\frac{1}{t^2}\right)}{\left(t^2+\frac{1}{t^2}\right)}dt - \int\frac{\left(1-\frac{1}{t^2}\right)}{\left(t^2+\frac{1}{t^2}\right)}dt$$

$$= -\int\frac{\left(1+\frac{1}{t^2}\right)}{\left[\left(t-\frac{1}{t}\right)^2+2\right]}dt - \int\frac{\left(1-\frac{1}{t^2}\right)}{\left[\left(t+\frac{1}{t}\right)^2-2\right]}dt$$

$$= -\int\frac{du}{[u^2+(\sqrt{2})^2]} - \int\frac{dv}{[v^2-(\sqrt{2})^2]}$$

[putting $\left(t-\frac{1}{t}\right)=u$ in the 1st integral, and $\left(t+\frac{1}{t}\right)=v$ in the 2nd]

$$= -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) - \frac{1}{2\sqrt{2}}\log\left|\frac{v-\sqrt{2}}{v+\sqrt{2}}\right| + C$$

$$= -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t-\frac{1}{t}}{\sqrt{2}}\right) - \frac{1}{2\sqrt{2}}\log\left|\frac{t+\frac{1}{t}-\sqrt{2}}{t+\frac{1}{t}+\sqrt{2}}\right| + C$$

$$= -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t^2-1}{\sqrt{2}t}\right) - \frac{1}{2\sqrt{2}}\log\left|\frac{t^2-\sqrt{2}t+1}{t^2+\sqrt{2}t+1}\right| + C$$

$$= -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\cot x-1}{\sqrt{2\cot x}}\right) - \frac{1}{2\sqrt{2}}\log\left|\frac{\cot x-\sqrt{2\cot x}+1}{\cot x+\sqrt{2\cot x}+1}\right| + C.$$

EXAMPLE 36 *Evaluate* $\int(\sqrt{\tan x}+\sqrt{\cot x})\,dx.$ **[CBSE 2010C, '13C]**

SOLUTION We have

$$\int(\sqrt{\tan x}+\sqrt{\cot x})\,dx$$

$$= \int\left(\sqrt{\tan x}+\frac{1}{\sqrt{\tan x}}\right)dx = \int\frac{(\tan x+1)}{\sqrt{\tan x}}\,dx$$

$$= \int\frac{(t^2+1)}{t}\cdot\frac{2t}{(1+t^4)}\,dt, \text{ where } \tan x = t^2 \Rightarrow x = \tan^{-1}t^2$$

$$\Rightarrow dx = \frac{2t}{(1+t^4)}\,dt$$

$$= 2\int\frac{(t^2+1)}{(t^4+1)}\,dt = 2\int\frac{\left(1+\frac{1}{t^2}\right)}{\left(t^2+\frac{1}{t^2}\right)}\,dt \text{ [on dividing num. and denom. by } t^2]$$

$$= 2\int\frac{\left(1+\frac{1}{t^2}\right)}{\left(t-\frac{1}{t}\right)^2+2}\,dt$$

$$= 2\int\frac{du}{(u^2+2)}, \text{ where } \left(t-\frac{1}{t}\right) = u \text{ and } \left(1+\frac{1}{t^2}\right)dt = du$$

$$= 2\cdot\frac{1}{\sqrt{2}}\tan^{-1}\frac{u}{\sqrt{2}} + C = \sqrt{2}\tan^{-1}\frac{\left(t-\frac{1}{t}\right)}{\sqrt{2}} + C \qquad \left[\because\ u = \left(t-\frac{1}{t}\right)\right]$$

$$= \sqrt{2}\tan^{-1}\frac{(t^2-1)}{(\sqrt{2}t)} + C = \sqrt{2}\tan^{-1}\left(\frac{\tan x-1}{\sqrt{2\tan x}}\right) + C \qquad [\because\ t^2 = \tan x].$$

EXAMPLE 37 *Evaluate* $\int\sqrt{\tan\theta}\,d\theta.$ **[CBSE 2006]**

SOLUTION Putting $\tan\theta = t^2$, we get $\theta = \tan^{-1}t^2 \Rightarrow d\theta = \frac{2t}{(1+t^4)}\,dt.$

$$\therefore\ I=\int t\cdot\frac{2t}{(1+t^4)}\,dt=\int\frac{2t^2}{(t^4+1)}\,dt$$

$$=\int\frac{(t^2+1)+(t^2-1)}{(t^4+1)}\,dt=\int\frac{(t^2+1)}{(t^4+1)}\,dt+\int\frac{(t^2-1)}{(t^4+1)}\,dt$$

$$=\int\frac{\left(1+\frac{1}{t^2}\right)}{\left(t^2+\frac{1}{t^2}\right)}\,dt+\int\frac{\left(1-\frac{1}{t^2}\right)}{\left(t^2+\frac{1}{t^2}\right)}\,dt$$

$$=\int\frac{\left(1+\frac{1}{t^2}\right)}{\left(t-\frac{1}{t}\right)^2+2}\,dt+\int\frac{\left(1-\frac{1}{t^2}\right)}{\left\{\left(t+\frac{1}{t}\right)^2-2\right\}}\,dt$$

$$=\int\frac{du}{(u^2+2)}+\int\frac{dv}{(v^2-2)},$$

where $\left(t-\frac{1}{t}\right)=u$ and $\left(t+\frac{1}{t}\right)=v$ in $I_1$ and $I_2$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\frac{u}{\sqrt{2}}+\frac{1}{2\sqrt{2}}\cdot\log\left|\frac{v-\sqrt{2}}{v+\sqrt{2}}\right|+C$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t-\frac{1}{t}}{\sqrt{2}}\right)+\frac{1}{2\sqrt{2}}\cdot\left|\frac{t+\frac{1}{t}-\sqrt{2}}{t+\frac{1}{t}+\sqrt{2}}\right|+C$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t^2-1}{\sqrt{2}t}\right)+\frac{1}{2\sqrt{2}}\cdot\left|\frac{t^2-\sqrt{2}t+1}{t^2+\sqrt{2}t+1}\right|+C,\ \text{where } t=\sqrt{\tan\theta}.$$

## EXERCISE 14A

***Evaluate:***

1. $\int\frac{dx}{(1-9x^2)}$
2. $\int\frac{dx}{(25-4x^2)}$
3. $\int\frac{dx}{(x^2+16)}$ **[CBSE 2011]**
4. $\int\frac{dx}{(4+9x^2)}$
5. $\int\frac{dx}{(50+2x^2)}$
6. $\int\frac{dx}{(16x^2-25)}$
7. $\int\frac{(x^2-1)}{(x^2+4)}\,dx$
8. $\int\frac{x^2}{(9+4x^2)}\,dx$
9. $\int\frac{e^x}{(e^{2x}+1)}\,dx$
10. $\int\frac{\sin x}{(1+\cos^2 x)}\,dx$
11. $\int\frac{\cos x}{(1+\sin^2 x)}\,dx$
12. $\int\frac{3x^5}{(1+x^{12})}\,dx$
13. $\int\frac{2x^3}{(4+x^8)}\,dx$
14. $\int\frac{dx}{(e^x+e^{-x})}$
15. $\int\frac{x}{(1-x^4)}\,dx$

16. $\int \frac{x^2}{(a^6 - x^6)} dx$ 17. $\int \frac{dx}{(x^2 + 4x + 8)}$ 18. $\int \frac{dx}{(4x^2 - 4x + 3)}$

19. $\int \frac{dx}{(2x^2 + x + 3)}$ 20. $\int \frac{dx}{(2x^2 - x - 1)}$ 21. $\int \frac{dx}{(3 - 2x - x^2)}$

22. $\int \frac{x}{(x^2 + 3x + 2)} dx$ 23. $\int \frac{(x - 3)}{(x^2 + 2x - 4)} dx$ 24. $\int \frac{(2x - 3)}{(x^2 + 3x - 18)} dx$

25. $\int \frac{x^2}{(x^2 + 6x - 3)} dx$ 26. $\int \frac{(2x - 1)}{(2x^2 + 2x + 1)} dx$ 27. $\int \frac{(1 - 3x)}{(3x^2 + 4x + 2)} dx$

28. $\int \frac{2x}{(2 + x - x^2)} dx$ 29. $\int \frac{dx}{(1 + \cos^2 x)}$ 30. $\int \frac{dx}{(2 + \sin^2 x)}$

31. $\int \frac{dx}{(a^2 \cos^2 x + b^2 \sin^2 x)}$ 32. $\int \frac{dx}{(\cos^2 x - 3 \sin^2 x)}$ 33. $\int \frac{dx}{(\sin^2 x - 4\cos^2 x)}$

34. $\int \frac{dx}{(\sin x \cos x + 2\cos^2 x)}$ 35. $\int \frac{\sin 2x}{(\sin^4 x + \cos^4 x)} dx$

36. $\int \frac{(2\sin 2\phi - \cos \phi)}{(6 - \cos^2 \phi - 4\sin \phi)} d\phi$ 37. $\int \frac{dx}{(\sin x - 2\cos x)(2\sin x + \cos x)}$

38. $\int \frac{(1 - x^2)}{(1 + x^4)} dx$ **[CBSE 2007]** 39. $\int \frac{(x^2 + 1)}{(x^4 + x^2 + 1)} dx$ 40. $\int \frac{dx}{(\sin^4 x + \cos^4 x)}$

## ***ANSWERS (EXERCISE 14A)***

**1.** $\frac{1}{6} \log \left| \frac{1 + 3x}{1 - 3x} \right| + C$ **2.** $\frac{1}{20} \log \left| \frac{5 + 2x}{5 - 2x} \right| + C$ **3.** $\frac{1}{4} \tan^{-1} \frac{x}{4} + C$

**4.** $\frac{1}{6} \tan^{-1} \left( \frac{3x}{2} \right) + C$ **5.** $\frac{1}{10} \tan^{-1} \frac{x}{5} + C$ **6.** $\frac{1}{40} \log \left| \frac{4x - 5}{4x + 5} \right| + C$

**7.** $x - \frac{5}{2} \tan^{-1} \frac{x}{2} + C$ **8.** $\frac{x}{4} - \frac{3}{8} \tan^{-1} \left( \frac{2x}{3} \right) + C$ **9.** $\tan^{-1}(e^x) + C$

**10.** $-\tan^{-1}(\cos x) + C$ **11.** $\tan^{-1}(\sin x) + C$ **12.** $\frac{1}{2} \tan^{-1}(x^6) + C$

**13.** $\frac{1}{4} \tan^{-1} \left( \frac{x^4}{2} \right) + C$ **14.** $\tan^{-1}(e^x) + C$ **15.** $\frac{1}{4} \log \left| \frac{1 + x^2}{1 - x^2} \right| + C$

**16.** $\frac{1}{6a^3} \log \left| \frac{a^3 + x^3}{a^3 - x^3} \right| + C$ **17.** $\frac{1}{2} \tan^{-1} \left( \frac{x + 2}{2} \right) + C$ **18.** $\frac{1}{2\sqrt{2}} \tan^{-1} \left( \frac{2x - 1}{\sqrt{2}} \right) + C$

**19.** $\frac{2}{\sqrt{23}} \tan^{-1} \left( \frac{4x + 1}{\sqrt{23}} \right) + C$ **20.** $\frac{1}{3} \log \left| \frac{2(x - 1)}{2x + 1} \right| + C$ **21.** $\frac{1}{4} \log \left| \frac{3 + x}{1 - x} \right| + C$

**22.** $\frac{1}{2} \log |x^2 + 3x + 2| - \frac{3}{2} \log \left| \frac{x + 1}{x + 2} \right| + C$

**23.** $\frac{1}{2}\log|x^2+2x-4| - \frac{2}{\sqrt{5}}\log\left|\frac{x+1-\sqrt{5}}{x+1+\sqrt{5}}\right| + C$

**24.** $\log|x^2+3x-18| - \frac{2}{3}\log\left|\frac{x-3}{x+6}\right| + C$

**25.** $x - 3\log|x^2+6x-3| + \frac{7\sqrt{3}}{4}\log\left|\frac{x+3-2\sqrt{3}}{x+3+2\sqrt{3}}\right| + C$

**26.** $\frac{1}{2}\log|2x^2+2x+1| - 2\tan^{-1}(2x+1) + C$

**27.** $-\frac{1}{2}\log|3x^2+4x+2| + \frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{3x+2}{\sqrt{2}}\right) + C$

**28.** $-\log|2+x-x^2| + \frac{1}{3}\log\left|\frac{1+x}{2-x}\right| + C$

**29.** $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right) + C$

**30.** $\frac{1}{\sqrt{6}}\tan^{-1}\left(\frac{\sqrt{3}\tan x}{\sqrt{2}}\right) + C$

**31.** $\frac{1}{ab}\tan^{-1}\left(\frac{b}{a}\tan x\right) + C$

**32.** $\frac{1}{2\sqrt{3}}\log\left|\frac{1+\sqrt{3}\tan x}{1-\sqrt{3}\tan x}\right| + C$

**33.** $\frac{1}{4}\log\left|\frac{\tan x-2}{\tan x+2}\right| + C$

**34.** $\log|\tan x+2| + C$

**35.** $\tan^{-1}(\tan^2 x) + C$

**36.** $2\log|\sin^2\phi - 4\sin\phi + 5| + 7\tan^{-1}(\sin\phi - 2) + C$

**37.** $\frac{1}{5}\log\left|\frac{\tan x-2}{2\tan x+1}\right| + C$

**38.** $\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2}x+x^2+1}{\sqrt{2}x-x^2-1}\right| + C$

**39.** $\frac{1}{\sqrt{3}}\tan^{-1}\frac{(x^2-1)}{\sqrt{3}x} + C$

**40.** $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan^2 x-1}{\sqrt{2}\tan x}\right) + C$

## ***HINTS TO SOME SELECTED QUESTIONS (EXERCISE 14A)***

1. On dividing, we get $\frac{(x^2-1)}{(x^2+4)} = \left\{1 - \frac{5}{(x^2+4)}\right\}$.

8. On dividing $x^2$ by $(4x^2+9)$, we get $\frac{x^2}{(9+4x^2)} = \frac{1}{4} - \frac{(9/4)}{4x^2+9}$.

   $\therefore\ I = \frac{1}{4}\int dx - \frac{9}{19}\cdot\int\frac{dx}{\left\{x^2+\left(\frac{3}{2}\right)^2\right\}}$.

9. Put $e^x = t$ and $e^x dx = dt$.

10. Put $\cos x = t$ and $-\sin x\, dx = dt$.

12. Put $x^6 = t$ and $6x^5 dx = dt$.

14. Multiply numerator and denominator by $e^x$ and put $e^x = t$.

15. Put $x^2 = t$ and $2x\,dx = dt$.

16. Put $x^3 = t$ and $3x^2 dx = dt$.

22. Let $x = A \cdot \frac{d}{dx}(x^2 + 3x + 2) + B$. Then, $x = A \cdot (2x + 3) + B$.

$\therefore \quad (2A = 1 \text{ and } 3A + B = 0) \Rightarrow \left(A = \frac{1}{2} \text{ and } B = \frac{-3}{2}\right)$.

23. Let $(x - 3) = A \cdot \frac{d}{dx}(x^2 + 2x - 4) + B \Rightarrow (x - 3) = A(2x + 2) + B$.

$\therefore \quad (2A = 1 \text{ and } 2A + B = -3) \Rightarrow \left(A = \frac{1}{2} \text{ and } B = -4\right)$.

24. Let $(2x - 3) = A \cdot \frac{d}{dx}(x^2 + 3x - 18) + B \Rightarrow (2x - 3) = A(2x + 3) + B$.

25. On dividing $x^2$ by $(x^2 + 6x - 3)$, we get

$$I = \int \left\{1 - \frac{6x - 3}{x^2 + 6x - 3}\right\} dx = x - 3\int \frac{(2x - 1)}{(x^2 + 6x - 3)}\,dx.$$

Let $(2x - 1) = A \cdot \frac{d}{dx}(x^2 + 6x - 3) + B \Rightarrow (2x - 1) = A(2x + 6) + B$.

29. On dividing num. and denom. by $\cos^2 x$, we get

$$I = \int \frac{\sec^2 x}{(2 + \tan^2 x)}\,dx = \int \frac{dt}{(2 + t^2)}, \text{ where } \tan x = t.$$

30. Divide num. and denom. by $\cos^2 x$.

31. Divide num. and denom. by $\cos^2 x$ and put $\tan x = t$.

32. Divide num. and denom. by $\cos^2 x$ and put $\tan x = t$.

33. Divide num. and denom. by $\cos^2 x$ and put $\tan x = t$.

34. Divide num. and denom. by $\cos^2 x$ and put $\tan x = t$.

35. On dividing num. and denom. by $\cos^4 x$ and putting $\tan x = t$, $\sec^2 x\,dx = dt$, we get

$I = \int \frac{2t}{(t^4 + 1)}\,dt$. Now put $t^2 = u$.

36. $I = \int \frac{(4\sin\phi - 1)\cos\phi}{(\sin^2\phi - 4\sin\phi + 5)}\,d\phi = \int \frac{(4t - 1)}{(t^2 - 4t + 5)}\,dt$, where $\sin\phi = t$.

Let $(4t - 1) = A \cdot \frac{d}{dt}(t^2 - 4t + 5) + B$.

37. $I = \int \frac{dx}{(2\sin^2 x - 3\sin x\cos x - 2\cos^2 x)}$.

Now, divide num. and denom. by $\cos^2 x$ and put $\tan x = t$.

38. $I = -\int \frac{\left(1 - \frac{1}{x^2}\right)}{\left(x^2 + \frac{1}{x^2}\right)}\,dx = -\int \frac{\left(1 - \frac{1}{x^2}\right)}{\left(x + \frac{1}{x}\right)^2 - 2}$

$$= \int \frac{-dt}{(t^2 - 2)}, \text{ where } \left(x + \frac{1}{x}\right) = t \text{ and } \left(1 - \frac{1}{x^2}\right) dx = dt$$

$$= \int \frac{dt}{(\sqrt{2})^2 - t^2} = \frac{1}{2\sqrt{2}} \log \left| \frac{\sqrt{2} + t}{\sqrt{2} - t} \right| + C$$

$$= \frac{1}{2\sqrt{2}} \log \left| \frac{\sqrt{2} + x + \frac{1}{x}}{\sqrt{2} - x - \frac{1}{x}} \right| + C = \frac{1}{2\sqrt{2}} \log \left| \frac{\sqrt{2}x + x^2 + 1}{\sqrt{2}x - x^2 - 1} \right| + C.$$

39. $\dfrac{1}{(\sin^4 x + \cos^4 x)} = \dfrac{\sec^4 x}{(\tan^4 x + 1)} = \dfrac{(1 + \tan^2 x)\sec^2 x}{(\tan^4 x + 1)}$.

Now, put $\tan x = t$.

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## Three More Special Integrals

THEOREM (i) $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C.$

(ii) $\displaystyle\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + C.$

(iii) $\displaystyle\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + C.$

PROOF (i) Put $x = a \sin\theta$ so that $dx = a\cos\theta\, d\theta$.

$$\therefore \quad \int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{a\cos\theta}{a\cos\theta} d\theta = \int d\theta = \theta + C = \sin^{-1}\frac{x}{a} + C.$$

Hence, $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C.$

(ii) Put $x = a\sec\theta$ so that $dx = a\sec\theta\tan\theta\, d\theta$.

$$\therefore \quad \int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{a\sec\theta\tan\theta}{a\tan\theta} d\theta$$

$$= \int \sec\theta\, d\theta = \log|\sec\theta + \tan\theta| + c$$

$$= \log\left|\sec\theta + \sqrt{\sec^2\theta - 1}\right| + c$$

$$= \log\left|\frac{x}{a} + \sqrt{\left(\frac{x^2}{a^2} - 1\right)}\right| + c = \log\left|\frac{x + \sqrt{x^2 - a^2}}{a}\right| + c$$

$$= \log\left|x + \sqrt{x^2 - a^2}\right| - \log a + c$$

$$= \log\left|x + \sqrt{x^2 - a^2}\right| + C \quad [\text{taking } -\log a + c = C].$$

Hence, $\displaystyle\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + C.$

(iii) Put $x = a\tan\theta$ so that $dx = a\sec^2\theta\, d\theta$.

$$\therefore \int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a\sec^2\theta}{a\sec\theta}\, d\theta$$

$$= \int \sec\theta\, d\theta = \log|\sec\theta + \tan\theta| + c$$

$$= \log\left|\sqrt{1+\tan^2\theta} + \tan\theta\right| + c$$

$$= \log\left|\sqrt{1+\frac{x^2}{a^2}} + \frac{x}{a}\right| + c$$

$$= \log\left|x+\sqrt{x^2+a^2}\right| - \log a + c$$

$$= \log\left|x+\sqrt{x^2+a^2}\right| + C \quad [\text{taking } -\log a + c = C].$$

Hence, $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + C.$

## SOLVED EXAMPLES

EXAMPLE 1 *Evaluate:*

*(i)* $\displaystyle\int \frac{dx}{\sqrt{9-25x^2}}$ *(ii)* $\displaystyle\int \frac{dx}{\sqrt{4x^2-9}}$ *(iii)* $\displaystyle\int \frac{dx}{\sqrt{16x^2+25}}$

SOLUTION We have

(i) $$\int \frac{dx}{\sqrt{9-25x^2}} = \frac{1}{5}\cdot\int \frac{dx}{\sqrt{\frac{9}{25}-x^2}} = \frac{1}{5}\cdot\int \frac{dx}{\sqrt{\left(\frac{3}{5}\right)^2 - x^2}}$$

$$= \frac{1}{5}\sin^{-1}\left(\frac{x}{3/5}\right) + C = \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right) + C.$$

(ii) $$\int \frac{dx}{\sqrt{4x^2-9}} = \frac{1}{2}\int \frac{dx}{\sqrt{x^2-\frac{9}{4}}} = \frac{1}{2}\int \frac{dx}{\sqrt{x^2-\left(\frac{3}{2}\right)^2}}$$

$$= \frac{1}{2}\log\left|x+\sqrt{x^2-\frac{9}{4}}\right| + C$$

$$= \frac{1}{2}\log\left|2x+\sqrt{4x^2-9}\right| + C.$$

(iii) $$\int \frac{dx}{\sqrt{16x^2+25}} = \frac{1}{4}\cdot\int \frac{dx}{\sqrt{x^2+\frac{25}{16}}} = \frac{1}{4}\cdot\int \frac{dx}{\sqrt{x^2+\left(\frac{5}{4}\right)^2}}$$

$$= \frac{1}{4}\cdot\log\left|x+\sqrt{x^2+\frac{25}{16}}\right| + C$$

$$= \frac{1}{4}\log\left|4x+\sqrt{16x^2+25}\right| + C.$$

EXAMPLE 2 *Evaluate* $\int \frac{dx}{\sqrt{15-8x^2}}\cdot$ **[CBSE 2002C]**

SOLUTION We have

$$\int \frac{dx}{\sqrt{15-8x^2}} = \frac{1}{\sqrt{8}}\cdot\int \frac{dx}{\sqrt{\frac{15}{8}-x^2}}$$

$$= \frac{1}{2\sqrt{2}}\sin^{-1}\left\{\frac{x}{\sqrt{\frac{15}{8}}}\right\} + C = \frac{1}{2\sqrt{2}}\sin^{-1}\left(\sqrt{\frac{8}{15}}x\right) + C.$$

EXAMPLE 3 *Evaluate* $\int \frac{\cos x}{\sqrt{4-\sin^2 x}}\,dx.$

SOLUTION Putting $\sin x = t$ and $\cos x\,dx = dt$, we get

$$\int \frac{\cos x}{\sqrt{4-\sin^2 x}}\,dx = \int \frac{dt}{\sqrt{4-t^2}} = \int \frac{dt}{\sqrt{2^2-t^2}}$$

$$= \sin^{-1}\frac{t}{2} + C = \sin^{-1}\left(\frac{\sin x}{2}\right) + C.$$

EXAMPLE 4 *Evaluate* $\int \frac{dx}{\sqrt{1-e^{2x}}}\cdot$

SOLUTION Multiplying the numerator and denominator by $e^{-x}$, we get

$$\int \frac{dx}{\sqrt{1-e^{2x}}} = \int \frac{e^{-x}\,dx}{\sqrt{e^{-2x}(1-e^{2x})}} = \int \frac{e^{-x}}{\sqrt{e^{-2x}-1}}\,dx$$

$$= -\int \frac{dt}{\sqrt{t^2-1}} \quad [\text{putting } e^{-x} = t]$$

$$= -\log\left|t + \sqrt{t^2-1}\right| + C$$

$$= -\log\left|e^{-x} + \sqrt{e^{-2x}-1}\right| + C.$$

EXAMPLE 5 *Evaluate* $\int \frac{2^x}{\sqrt{1-4^x}}\,dx.$

SOLUTION Putting $2^x = t$ and $(2^x \log 2)\,dx = dt$, we get

$$\int \frac{2^x}{\sqrt{1-4^x}}\,dx = \frac{1}{(\log 2)}\cdot\int \frac{dt}{\sqrt{1-t^2}}$$

$$= \frac{1}{(\log 2)}\cdot\sin^{-1} t + C = \frac{1}{(\log 2)}\cdot\sin^{-1}(2^x) + C.$$

EXAMPLE 6 *Evaluate* $\int \frac{x^2}{\sqrt{x^6-1}}\,dx.$

SOLUTION Putting $x^3 = t$ and $x^2 dx = \frac{1}{3}\,dt$, we get

$$\int \frac{x^2}{\sqrt{x^6-1}}\,dx = \frac{1}{3}\int \frac{dt}{\sqrt{t^2-1}} = \frac{1}{3}\log\left|t+\sqrt{t^2-1}\right|+C$$
$$= \frac{1}{3}\log\left|x^3+\sqrt{x^6-1}\right|+C.$$

EXAMPLE 7 *Evaluate:*

*(i)* $\int \frac{\sin x}{\sqrt{4\cos^2 x-1}}\,dx$ *(ii)* $\int \frac{\sec^2 x}{\sqrt{\tan^2 x-4}}\,dx$

SOLUTION (i) Putting $\cos x = t$ and $-\sin x\,dx = dt$, we get

$$\int \frac{\sin x}{\sqrt{4\cos^2 x-1}}\,dx = \int \frac{-dt}{\sqrt{4t^2-1}} = -\frac{1}{2}\int \frac{dt}{\sqrt{t^2-(1/2)^2}}$$
$$= -\frac{1}{2}\cdot\log\left|t+\sqrt{t^2-\frac{1}{4}}\right|+C$$
$$= -\frac{1}{2}\log\left|2t+\sqrt{4t^2-1}\right|+C$$
$$= -\frac{1}{2}\log\left|2\cos x+\sqrt{4\cos^2 x-1}\right|+C.$$

(ii) Putting $\tan x = t$ and $\sec^2 x\,dx = dt$, we get

$$\int \frac{\sec^2 x}{\sqrt{\tan^2 x-4}}\,dx = \int \frac{dt}{\sqrt{t^2-4}} = \log\left|t+\sqrt{t^2-4}\right|+C$$
$$= \log\left|\tan x+\sqrt{\tan^2 x-4}\right|+C.$$

EXAMPLE 8 *Evaluate* $\int \frac{x^2}{\sqrt{x^6+a^6}}\,dx.$

SOLUTION Putting $x^3 = t$ and $x^2dx = \frac{1}{3}\,dt$, we get

$$\int \frac{x^2}{\sqrt{x^6+a^6}}\,dx = \frac{1}{3}\int \frac{dt}{\sqrt{t^2+(a^3)^2}} = \frac{1}{3}\log\left|t+\sqrt{t^2+a^6}\right|+C$$
$$= \frac{1}{3}\log\left|x^3+\sqrt{x^6+a^6}\right|+C.$$

EXAMPLE 9 *Evaluate* $\int \frac{\sec^2 x}{\sqrt{16+\tan^2 x}}\,dx.$

SOLUTION Putting $\tan x = t$ and $\sec^2 x\,dx = dt$, we get

$$\int \frac{\sec^2 x}{\sqrt{16+\tan^2 x}}\,dx = \int \frac{dt}{\sqrt{16+t^2}} = \log\left|t+\sqrt{t^2+16}\right|+C$$
$$= \log\left|\tan x+\sqrt{\tan^2 x+16}\right|+C.$$

EXAMPLE 10 *Evaluate* $\int \sqrt{\frac{1-x}{1+x}}\,dx.$ **[CBSE 2006]**

SOLUTION We have

$$\int \sqrt{\frac{1-x}{1+x}}\, dx = \int \left\{ \frac{\sqrt{1-x}}{\sqrt{1+x}} \times \frac{\sqrt{1-x}}{\sqrt{1-x}} \right\} dx$$

$$= \int \frac{(1-x)}{\sqrt{1-x^2}}\, dx = \int \frac{dx}{\sqrt{1-x^2}} - \int \frac{x}{\sqrt{1-x^2}}\, dx$$

$$= \sin^{-1} x + \frac{1}{2} \cdot \int \frac{(-2x)}{\sqrt{1-x^2}}\, dx$$

$$= \sin^{-1} x + \frac{1}{2} \int \frac{dt}{\sqrt{t}}, \text{ where } (1-x^2) = t \text{ and } (-2x)\, dx = dt$$

$$= \sin^{-1} x + \frac{1}{2} \int t^{-1/2}\, dt = \sin^{-1} x + \frac{1}{2} \cdot \frac{t^{1/2}}{(1/2)} + C$$

$$= \sin^{-1} x + \sqrt{1-x^2} + C.$$

**Integrals of the form** $\int \frac{dx}{\sqrt{(ax^2 + bx + c)}}$.

**METHOD** Put $(ax^2 + bx + c)$ in the form $a\{(x + \alpha)^2 \pm \beta^2\}$ and then integrate.

EXAMPLE 11 *Evaluate* $\int \frac{dx}{\sqrt{x^2 - 3x + 2}}$. **[CBSE 2006]**

SOLUTION We have

$$\int \frac{dx}{\sqrt{x^2 - 3x + 2}} = \int \frac{dx}{\sqrt{\left(x^2 - 3x + \frac{9}{4}\right) - \frac{1}{4}}} = \int \frac{dx}{\sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}$$

$$= \int \frac{dz}{\sqrt{z^2 - \left(\frac{1}{2}\right)^2}}, \text{ where } \left(x - \frac{3}{2}\right) = z$$

$$= \log \left| z + \sqrt{z^2 - \frac{1}{4}} \right| + C$$

$$\left[ \because \int \frac{dz}{\sqrt{z^2 - a^2}} = \log \left| z + \sqrt{z^2 - a^2} \right| + C \right]$$

$$= \log \left| \left(x - \frac{3}{2}\right) + \sqrt{x^2 - 3x + 2} \right| + C.$$

EXAMPLE 12 *Evaluate* $\int \frac{dx}{\sqrt{5x^2 - 2x}}$.

SOLUTION We have

$$\int \frac{dx}{\sqrt{5x^2 - 2x}} = \frac{1}{\sqrt{5}} \cdot \int \frac{dx}{\sqrt{x^2 - \frac{2}{5}x}} = \frac{1}{\sqrt{5}} \cdot \int \frac{dx}{\sqrt{x^2 - \frac{2}{5}x + \left(\frac{1}{5}\right)^2 - \left(\frac{1}{5}\right)^2}}$$

$$= \frac{1}{\sqrt{5}} \cdot \int \frac{dx}{\sqrt{\left(x - \frac{1}{5}\right)^2 - \left(\frac{1}{5}\right)^2}} = \frac{1}{\sqrt{5}} \cdot \int \frac{dt}{\sqrt{t^2 - \left(\frac{1}{5}\right)^2}},$$

where $\left(x - \frac{1}{5}\right) = t$

$$= \frac{1}{\sqrt{5}} \cdot \log \left| t + \sqrt{t^2 - \left(\frac{1}{5}\right)^2} \right| + C$$

$$= \frac{1}{\sqrt{5}} \log \left| \left(x - \frac{1}{5}\right) + \sqrt{x^2 - \frac{2x}{5}} \right| + C.$$

EXAMPLE 13 *Evaluate* $\displaystyle\int \frac{\cos x}{\sqrt{\sin^2 x - 2\sin x - 3}}\, dx.$ **[CBSE 2006C]**

SOLUTION Putting $\sin x = t$ and $\cos x\, dx = dt$, we get

$$\int \frac{\cos x}{\sqrt{\sin^2 x - 2\sin x - 3}}\, dx = \int \frac{dt}{\sqrt{t^2 - 2t - 3}} = \int \frac{dt}{\sqrt{(t-1)^2 - 2^2}}$$

$$= \log \left| (t-1) + \sqrt{(t-1)^2 - 2^2} \right| + C$$

$$= \log \left| (\sin x - 1) + \sqrt{\sin^2 x - 2\sin x - 3} \right| + C.$$

EXAMPLE 14 *Evaluate* $\displaystyle\int \frac{e^x}{\sqrt{5 - 4e^x - e^{2x}}}\, dx.$ **[CBSE 2005C, '09]**

SOLUTION Putting $e^x = t$ and $e^x dx = dt$, we get

$$\int \frac{e^x}{\sqrt{5 - 4e^x - e^{2x}}}\, dx = \int \frac{dt}{\sqrt{5 - 4t - t^2}} = \int \frac{dt}{\sqrt{5 - (t^2 + 4t + 4) + 4}}$$

$$= \int \frac{dt}{\sqrt{9 - (t+2)^2}} = \int \frac{dt}{\sqrt{3^2 - (t+2)^2}}$$

$$= \int \frac{dz}{\sqrt{3^2 - z^2}}, \text{ where } (t+2) = z$$

$$= \sin^{-1} \frac{z}{3} + C = \sin^{-1} \frac{(t+2)}{3} + C$$

$$= \sin^{-1} \frac{(e^x + 2)}{3} + C.$$

EXAMPLE 15 *Evaluate* $\displaystyle\int \frac{dx}{\sqrt{2 - 4x + x^2}}.$

SOLUTION We have

$$\int \frac{dx}{\sqrt{2 - 4x + x^2}} = \int \frac{dx}{\sqrt{x^2 - 4x + 4 - 2}} = \int \frac{dx}{\sqrt{(x-2)^2 - (\sqrt{2})^2}}$$

$$= \log \left| (x-2) + \sqrt{(x-2)^2 - 2} \right| + C$$

$$= \log \left| x - 2 + \sqrt{x^2 - 4x + 2} \right| + C.$$

EXAMPLE 16 *Evaluate* $\int \frac{dx}{\sqrt{3x^2+6x+12}}$. **[CBSE 2004C]**

SOLUTION We have

$$\int \frac{dx}{\sqrt{3x^2+6x+12}} = \frac{1}{\sqrt{3}} \cdot \int \frac{dx}{\sqrt{x^2+2x+4}}$$
$$= \frac{1}{\sqrt{3}} \cdot \int \frac{dx}{\sqrt{(x+1)^2+(\sqrt{3})^2}}$$
$$= \frac{1}{\sqrt{3}} \cdot \int \frac{dt}{\sqrt{t^2+(\sqrt{3})^2}}, \text{ where } (x+1)=t$$
$$= \frac{1}{\sqrt{3}} \log \left| t + \sqrt{t^2+3} \right| + C$$
$$= \frac{1}{\sqrt{3}} \log \left| (x+1) + \sqrt{x^2+2x+4} \right| + C.$$

EXAMPLE 17 *Evaluate* $\int \frac{dx}{\sqrt{8+3x-x^2}}$.

SOLUTION We have

$$\int \frac{dx}{\sqrt{8+3x-x^2}} = \int \frac{dx}{\sqrt{8-(x^2-3x)}} = \int \frac{dx}{\sqrt{8-\left(x^2-3x+\frac{9}{4}\right)+\frac{9}{4}}}$$
$$= \int \frac{dx}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}} = \sin^{-1}\left\{\frac{\left(x-\frac{3}{2}\right)}{\left(\frac{\sqrt{41}}{2}\right)}\right\}+C$$
$$= \sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right) + C.$$

EXAMPLE 18 *Evaluate* $\int \frac{dx}{\sqrt{2x-x^2}}$.

SOLUTION We have

$$\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dx}{\sqrt{1-(x^2-2x+1)}}$$
$$= \int \frac{dx}{\sqrt{1-(x-1)^2}} = \sin^{-1}(x-1) + C.$$

EXAMPLE 19 *Evaluate* $\int \frac{dx}{\sqrt{x(1-2x)}}$.

SOLUTION We have

$$\int \frac{dx}{\sqrt{x(1-2x)}} = \int \frac{dx}{\sqrt{x-2x^2}}$$
$$= \frac{1}{\sqrt{2}} \cdot \int \frac{dx}{\sqrt{\frac{x}{2}-x^2}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{-\left(x^2-\frac{x}{2}+\frac{1}{16}\right)+\frac{1}{16}}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\frac{1}{16} - \left\{x^2 - \frac{x}{2} + \frac{1}{16}\right\}}} = \frac{1}{\sqrt{2}} \cdot \int \frac{dx}{\sqrt{\left(\frac{1}{4}\right)^2 - \left(x - \frac{1}{4}\right)^2}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{\left(\frac{1}{4}\right)^2 - t^2}}, \text{ where } \left(x - \frac{1}{4}\right) = t$$

$$= \frac{1}{\sqrt{2}} \sin^{-1} \frac{t}{(1/4)} + C = \frac{1}{\sqrt{2}} \sin^{-1}(4t) + C = \frac{1}{\sqrt{2}} \sin^{-1} 4\left(x - \frac{1}{4}\right) + C$$

$$= \frac{1}{\sqrt{2}} \sin^{-1}(4x - 1) + C.$$

**Integrals of the form** $\int \frac{(px+q)}{\sqrt{(ax^2+bx+c)}}\,dx.$

METHOD Let $(px+q) = A \cdot \frac{d}{dx}(ax^2 + bx + c) + B.$

Now, the value of the integral can be obtained easily.

EXAMPLE 20 *Evaluate* $\int \frac{(5x+3)}{\sqrt{x^2+4x+10}}\,dx.$ **[CBSE 2011, 12]**

SOLUTION Let $(5x+3) = A \cdot \frac{d}{dx}(x^2 + 4x + 10) + B.$

Then, $(5x+3) = A(2x+4) + B.$

Comparing the coefficients of like powers of $x$, we get

$$(2A = 5 \text{ and } 4A + B = 3) \Rightarrow \left(A = \frac{5}{2} \text{ and } B = -7\right)$$

$$\therefore \int \frac{(5x+3)}{\sqrt{x^2+4x+10}}\,dx = \int \left\{\frac{\frac{5}{2}\cdot(2x+4) - 7}{\sqrt{x^2+4x+10}}\right\} dx$$

$$= \frac{5}{2} \cdot \int \frac{(2x+4)}{\sqrt{x^2+4x+10}}\,dx - 7\int \frac{dx}{\sqrt{x^2+4x+10}}$$

$$= \frac{5}{2} \cdot \int \frac{1}{\sqrt{t}}\,dt - 7\int \frac{dx}{\sqrt{(x+2)^2 + (\sqrt{6})^2}}, \text{ where } t = x^2 + 4x + 10$$

$$= \left(\frac{5}{2} \times 2\sqrt{t}\right) - 7\log\left|(x+2) + \sqrt{x^2+4x+10}\right| + C$$

$$= 5\sqrt{x^2+4x+10} - 7\log\left|(x+2) + \sqrt{x^2+4x+10}\right| + C.$$

EXAMPLE 21 *Evaluate* $\int \frac{(x+1)}{\sqrt{4+5x-x^2}}\,dx.$

SOLUTION Let $(x+1) = A \cdot \frac{d}{dx}(4 + 5x - x^2) + B.$ Then,

$$(x+1) = A(5-2x) + B.$$

Comparing the coefficients of like powers of $x$, we get

$$(-2A = 1 \text{ and } 5A + B = 1) \Rightarrow \left(A = -\frac{1}{2} \text{ and } B = \frac{7}{2}\right).$$

$$\therefore \quad (x+1) = -\frac{1}{2}(5-2x) + \frac{7}{2}.$$

$$\therefore \quad \int \frac{(x+1)}{\sqrt{4+5x-x^2}}\,dx = \int \frac{-\frac{1}{2}(5-2x)+\frac{7}{2}}{\sqrt{4+5x-x^2}}\,dx$$

$$= -\frac{1}{2}\int \frac{(5-2x)}{\sqrt{4+5x-x^2}}\,dx + \frac{7}{2}\int \frac{1}{\sqrt{4+5x-x^2}}\,dx$$

$$= -\frac{1}{2}\int \frac{1}{\sqrt{t}}\,dt + \frac{7}{2}\cdot\int \frac{1}{\sqrt{4-(x^2-5x)}}\,dx, \text{ where } t = 4+5x-x^2$$

$$= -\frac{1}{2}\cdot 2\sqrt{t} + \frac{7}{2}\cdot\int \frac{dx}{\sqrt{4-\left(x^2-5x+\frac{25}{4}\right)+\frac{25}{4}}}$$

$$= -\sqrt{t} + \frac{7}{2}\cdot\int \frac{dx}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2 - \left(x-\frac{5}{2}\right)^2}}$$

$$= -\sqrt{4+5x-x^2} + \frac{7}{2}\cdot\sin^{-1}\frac{\left(x-\frac{5}{2}\right)}{\left(\frac{\sqrt{41}}{2}\right)} + C$$

$$= -\sqrt{4+5x-x^2} + \frac{7}{2}\sin^{-1}\left(\frac{2x-5}{\sqrt{41}}\right) + C.$$

EXAMPLE 22 *Evaluate* $\int \frac{(x+3)}{\sqrt{5-4x-x^2}}\,dx.$

SOLUTION Let $(x+3) = A\cdot\frac{d}{dx}(5-4x-x^2) + B$. Then,

$$(x+3) = A(-4-2x) + B.$$

Comparing the coefficients of like powers of $x$, we get

$$(-2A = 1 \text{ and } -4A + B = 3) \Rightarrow \left(A = \frac{-1}{2}, B = 1\right).$$

$$\therefore \quad \int \frac{(x+3)}{\sqrt{5-4x-x^2}}\,dx = -\frac{1}{2}\cdot\int \frac{(-4-2x)}{\sqrt{5-4x-x^2}}\,dx + \int \frac{dx}{\sqrt{5-4x-x^2}}$$

$$= -\frac{1}{2}\int \frac{dt}{\sqrt{t}} + \int \frac{dx}{\sqrt{5-(x^2+4x+4)+4}},$$

where $(5-4x-x^2) = t$

$$= -\frac{1}{2}\cdot\frac{t^{1/2}}{(1/2)} + \int \frac{dx}{\sqrt{3^2-(x+2)^2}}$$

$$= -\sqrt{t} + \sin^{-1}\frac{(x+2)}{3} + C$$

$$= -\sqrt{5-4x-x^2} + \sin^{-1}\frac{(x+2)}{3} + C.$$

## EXERCISE 14B

***Evaluate:***

1. $\int \frac{dx}{\sqrt{16-x^2}}$
2. $\int \frac{dx}{\sqrt{1-9x^2}}$
3. $\int \frac{dx}{\sqrt{15-8x^2}}$
4. $\int \frac{dx}{\sqrt{x^2-4}}$
5. $\int \frac{dx}{\sqrt{4x^2-1}}$
6. $\frac{dx}{\sqrt{9x^2-7}}$
7. $\int \frac{dx}{\sqrt{x^2+9}}$
8. $\int \frac{dx}{\sqrt{1+4x^2}}$
9. $\int \frac{dx}{\sqrt{9+4x^2}}$
10. $\int \frac{x}{\sqrt{9-x^4}}\,dx$
11. $\int \frac{3x^2}{\sqrt{9-16x^6}}\,dx$
12. $\int \frac{\sec^2 x}{\sqrt{16+\tan^2 x}}\,dx$
13. $\int \frac{\sin x}{\sqrt{4+\cos^2 x}}\,dx$
14. $\int \frac{\cos x}{\sqrt{9\sin^2 x-1}}\,dx$
15. $\int \frac{e^x}{\sqrt{4+e^{2x}}}\,dx$
16. $\int \frac{2e^x}{\sqrt{4-e^{2x}}}\,dx$
17. $\int \frac{dx}{\sqrt{1-e^x}}$
18. $\int \sqrt{\frac{a-x}{a+x}}\,dx$
19. $\int \frac{dx}{\sqrt{x^2+6x+5}}$ **[CBSE 2001C]**
20. $\int \frac{dx}{\sqrt{(2-x)^2+1}}$ **[CBSE 2001C]**
21. $\int \frac{dx}{\sqrt{(x-3)^2-1}}$
22. $\int \frac{dx}{\sqrt{x^2-6x+10}}$
23. $\int \frac{dx}{\sqrt{2+2x-x^2}}$
24. $\int \frac{dx}{\sqrt{8-4x-2x^2}}$ **[CBSE 2003C]**
25. $\int \frac{dx}{\sqrt{16-6x-x^2}}$
26. $\int \frac{dx}{\sqrt{7-6x-x^2}}$ **[CBSE 2002]**
27. $\int \frac{dx}{\sqrt{x-x^2}}$
28. $\int \frac{dx}{\sqrt{8+2x-x^2}}$ **[CBSE 2002]**
29. $\int \frac{dx}{\sqrt{x^2-3x+2}}$
30. $\int \frac{dx}{\sqrt{2x^2+3x-2}}$
31. $\int \frac{dx}{\sqrt{2x^2+4x+6}}$ **[CBSE 2004C]**
32. $\int \frac{dx}{\sqrt{1+2x-3x^2}}$

33. $\int \frac{dx}{\sqrt{x}\,\sqrt{5-x}}$

34. $\int \frac{dx}{\sqrt{3+4x-2x^2}}$

35. $\int \frac{x^2}{\sqrt{x^6+2x^3+3}}\,dx$

36. $\int \frac{(2x+3)}{\sqrt{x^2+x+1}}\,dx$

37. $\int \frac{(5x+3)}{\sqrt{x^2+4x+10}}\,dx$ **[CBSE 2012C]**

38. $\int \frac{(4x+3)}{\sqrt{2x^2+2x-3}}$ **[CBSE 2001]**

39. $\int \frac{(3-2x)}{\sqrt{2+x-x^2}}\,dx$

40. $\int \frac{(x+2)}{\sqrt{x^2+2x-1}}\,dx$

41. $\int \frac{(3x+1)}{\sqrt{5-2x-x^2}}\,dx$

42. $\int \frac{(6x+5)}{\sqrt{6+x-2x^2}}\,dx$

43. $\int \sqrt{\frac{1+x}{x}}\,dx$

44. $\int \frac{(x+2)}{\sqrt{x^2+5x+6}}\,dx$ **[CBSE 2014]**

## ***ANSWERS (EXERCISE 14B)***

**1.** $\sin^{-1}\frac{x}{4}+C$ **2.** $\frac{1}{3}\sin^{-1}3x+C$ **3.** $\frac{1}{2\sqrt{2}}\sin^{-1}\sqrt{\frac{8}{15}}\,x+C$

**4.** $\log\left|x+\sqrt{x^2-4}\right|+C$ **5.** $\frac{1}{2}\log\left|2x+\sqrt{4x^2-1}\right|+C$

**6.** $\frac{1}{3}\log\left|3x+\sqrt{9x^2-7}\right|+C$ **7.** $\log\left|x+\sqrt{x^2+9}\right|+C$

**8.** $\frac{1}{2}\log\left|2x+\sqrt{4x^2+1}\right|+C$ **9.** $\frac{1}{2}\log\left|2x+\sqrt{9+4x^2}\right|+C$

**10.** $\frac{1}{2}\sin^{-1}\left(\frac{x^2}{3}\right)+C$ **11.** $\frac{1}{4}\sin^{-1}\left(\frac{4x^3}{3}\right)+C$

**12.** $\log\left|\tan x+\sqrt{16+\tan^2 x}\right|+C$ **13.** $-\log\left|\cos x+\sqrt{4+\cos^2 x}\right|+C$

**14.** $\frac{1}{3}\log\left|3\sin x+\sqrt{9\sin^2 x-1}\right|+C$ **15.** $\log\left|e^x+\sqrt{4+e^{2x}}\right|+C$

**16.** $2\sin^{-1}\left(\frac{e^x}{2}\right)+C$ **17.** $-2\log\left|e^{-x/2}+\sqrt{e^{-x}-1}\right|+C$

**18.** $a\sin^{-1}\frac{x}{a}+\sqrt{a^2-x^2}+C$ **19.** $\log\left|(x+3)+\sqrt{x^2+6x+5}\right|+C$

**20.** $-\log\left|(2-x)+\sqrt{x^2-4x+5}\right|+C$ **21.** $\log\left|(x-3)+\sqrt{x^2-6x+8}\right|+C$

**22.** $\log\left|(x-3)+\sqrt{x^2-6x+10}\right|+C$ **23.** $\sin^{-1}\left(\frac{x-1}{\sqrt{3}}\right)+C$

**24.** $\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x+1}{\sqrt{5}}\right)+C$ **25.** $\sin^{-1}\left(\frac{x+3}{5}\right)+C$

**26.** $\sin^{-1}\left(\frac{x+3}{4}\right)+C$

**27.** $\sin^{-1}(2x-1)+C$

**28.** $\sin^{-1}\left(\frac{x-1}{3}\right)+C$

**29.** $\log\left|\left(x-\frac{3}{2}\right)+\sqrt{x^2-3x+2}\right|+C$

**30.** $\frac{1}{\sqrt{2}}\log\left|x+\frac{3}{4}+\sqrt{x^2+\frac{3}{2}x-1}\right|+C$

**31.** $\frac{1}{\sqrt{2}}\log\left|(x+1)+\sqrt{x^2+2x+3}\right|+C$

**32.** $\frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{3x-1}{2}\right)+C$

**33.** $\sin^{-1}\left(\frac{2x-5}{5}\right)+C$

**34.** $\frac{1}{\sqrt{2}}\sin^{-1}\left\{\frac{\sqrt{2}(x-1)}{\sqrt{5}}\right\}+C$

**35.** $\frac{1}{3}\log\left|(x^3+1)+\sqrt{x^6+2x^3+3}\right|+C$

**36.** $2\sqrt{x^2+x+1}+2\log\left|x+\frac{1}{2}+\sqrt{x^2+x+1}\right|+C$

**37.** $5\sqrt{x^2+4x+10}-7\log\left|(x+2)+\sqrt{x^2+4x+10}\right|+C$

**38.** $2\sqrt{2x^2+2x-3}+\frac{1}{\sqrt{2}}\log\left|x+\frac{1}{2}+\sqrt{x^2+x-\frac{3}{2}}\right|+C$

**39.** $2\sqrt{2+x-x^2}+2\sin^{-1}\left(\frac{2x-1}{3}\right)+C$

**40.** $\sqrt{x^2+2x-1}+\log\left|(x+1)+\sqrt{x^2+2x-1}\right|+C$

**41.** $-3\sqrt{5-2x-x^2}-2\sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right)+C$

**42.** $-3\sqrt{6+x-2x^2}+\frac{13}{2\sqrt{2}}\sin^{-1}\left(\frac{4x-1}{7}\right)+C$

**43.** $\sqrt{x+x^2}+\frac{1}{2}\log\left|x+\frac{1}{2}+\sqrt{x+x^2}\right|+C$

**44.** $\sqrt{x^2+5x+6}-\frac{1}{2}\log\left|\left(x+\frac{5}{2}\right)+\sqrt{x^2+5x+6}\right|+C$

## HINTS TO SOME SELECTED QUESTIONS (EXERCISE 14B)

10. Put $x^2=t$ and $x\,dx=\frac{1}{2}\,dt$.
11. Put $x^3=t$ and $3x^2dx=dt$.
12. Put $\tan x=t$ and $\sec^2 x\,dx=dt$.
13. Put $\cos x=t$ and $\sin x\,dx=-\,dt$.
14. Put $\sin x=t$ and $\cos x\,dx=dt$.
15. Put $e^x=t$ and $e^x dx=dt$.
16. Put $e^x=t$ and $e^x dx=dt$.

17. Multiplying the numerator and denominator by $e^{-x/2}$, we get

$$I = \int \frac{e^{-x/2}}{\sqrt{e^{-x} - 1}}\,dx. \text{ Now, put } e^{-x/2} = t \text{ and } e^{-x/2} \cdot \left(\frac{-1}{2}\right) dx = dt.$$

18. $$\int \sqrt{\frac{a-x}{a+x}}\,dx = \int \left\{ \frac{\sqrt{a-x}}{\sqrt{a+x}} \times \frac{\sqrt{a-x}}{\sqrt{a-x}} \right\} dx = \int \frac{(a-x)}{\sqrt{a^2 - x^2}}\,dx$$

$$= a\int \frac{dx}{\sqrt{a^2 - x^2}} + \frac{1}{2} \cdot \int \frac{(-2x)}{\sqrt{a^2 - x^2}}\,dx = a\sin^{-1}\frac{x}{a} + \frac{1}{2}\int \frac{1}{\sqrt{t}}\,dt, \text{ where } (a^2 - x^2) = t$$

$$= a\sin^{-1}\frac{x}{a} + \frac{1}{2} \times 2\sqrt{t} + C = a\sin^{-1}\frac{x}{a} + \sqrt{a^2 - x^2} + C.$$

20. Put $(2 - x) = t$ and $dx = -\,dt$.

25. $(16 - 6x - x^2) = \{16 - (x^2 + 6x + 9) + 9\} = [25 - (x + 3)^2]$.

27. $$\sqrt{(x - x^2)} = \sqrt{-\left(x^2 - x + \frac{1}{4}\right) + \frac{1}{4}} = \sqrt{\left(\frac{1}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2}\,.$$

32. $$I = \frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{-x^2 + \frac{2}{3}x + \frac{1}{3}}} = \frac{1}{\sqrt{3}} \cdot \int \frac{dx}{\sqrt{\frac{1}{3} - \left(x^2 - \frac{2}{3}x + \frac{1}{9}\right) + \frac{1}{9}}}$$

$$= \frac{1}{\sqrt{3}} \cdot \int \frac{dx}{\sqrt{\left(\frac{2}{3}\right)^2 - \left(x - \frac{1}{3}\right)^2}}\,.$$

33. $$I = \int \frac{dx}{\sqrt{5x - x^2}} = \int \frac{dx}{\sqrt{-\left(x^2 - 5x + \frac{25}{4}\right) + \frac{25}{4}}} = \int \frac{dx}{\sqrt{\left(\frac{5}{2}\right)^2 - \left(x - \frac{5}{2}\right)^2}}\,.$$

34. $$I = \frac{1}{\sqrt{2}} \cdot \int \frac{dx}{\sqrt{-x^2 + 2x + 3}} = \frac{1}{\sqrt{2}} \cdot \int \frac{dx}{\{3 - (x^2 - 2x + 1) + 1\}} = \frac{1}{\sqrt{2}} \cdot \int \frac{dx}{\sqrt{2^2 - (x + 1)^2}}\,.$$

35. Put $x^3 = t$ and $3x^2 dx = dt$.

43. $$I = \int \left( \frac{\sqrt{1+x}}{\sqrt{x}} \times \frac{\sqrt{1+x}}{\sqrt{1+x}} \right) dx = \int \frac{(1+x)}{\sqrt{x(1+x)}}\,dx = \int \frac{(x+1)}{\sqrt{x^2 + x}}\,dx.$$

44. Let $(x + 2) = A \cdot \dfrac{d}{dx}(x^2 + 5x + 6) + B = A(2x + 5) + B$

$\therefore\ (2A = 1 \text{ and } 5A + B = 2) \Rightarrow A = \dfrac{1}{2} \text{ and } B = \dfrac{-1}{2}\cdot$

$$\therefore\ I = \int \frac{\left\{\frac{1}{2}(2x + 5) - \frac{1}{2}\right\}}{\sqrt{x^2 + 5x + 6}}\,dx = \frac{1}{2}\int \frac{(2x + 5)}{\sqrt{x^2 + 5x + 6}}\,dx - \frac{1}{2}\int \frac{dx}{\sqrt{\left(x + \frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}$$

$$= \sqrt{x^2 + 5x + 6} - \frac{1}{2}\log\left|\left(x + \frac{5}{2}\right) + \sqrt{x^2 + 5x + 6}\right| + C.$$

## Three More Special Integrals

THEOREM

(i) $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C.$

(ii) $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C.$

(iii) $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + C.$

PROOF (i) Integrating by parts, taking 1 as the second function, we get

$$I = \int \sqrt{a^2 - x^2}\, dx = \int (\sqrt{a^2 - x^2} \cdot 1)\, dx$$

$$= (\sqrt{a^2 - x^2}) \cdot x - \int \frac{1}{2}(a^2 - x^2)^{-1/2}(-2x) \cdot x\, dx$$

$$= x(\sqrt{a^2 - x^2}) + \int \frac{x^2}{\sqrt{a^2 - x^2}}\, dx = x(\sqrt{a^2 - x^2}) + \int \frac{a^2 - (a^2 - x^2)}{\sqrt{a^2 - x^2}}\, dx$$

$$= x(\sqrt{a^2 - x^2}) + a^2 \int \frac{dx}{\sqrt{a^2 - x^2}} - \int \sqrt{a^2 - x^2}\, dx$$

$$= x(\sqrt{a^2 - x^2}) + a^2 \sin^{-1}\frac{x}{a} - I + c.$$

$$\therefore \quad 2I = x(\sqrt{a^2 - x^2}) + a^2 \sin^{-1}\frac{x}{a} + c$$

$$\Rightarrow I = \frac{x}{2}(\sqrt{a^2 - x^2}) + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C \quad \left[\text{taking } \frac{c}{2} = C\right].$$

Hence, $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}(\sqrt{a^2 - x^2}) + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C.$

(ii) Integrating by parts, taking 1 as the second function, we get

$$I = \int \sqrt{x^2 - a^2}\, dx = \int (\sqrt{x^2 - a^2} \cdot 1)\, dx$$

$$= (\sqrt{x^2 - a^2}) \cdot x - \int \frac{1}{2}(x^2 - a^2)^{-1/2}(2x) \cdot x\, dx$$

$$= x(\sqrt{x^2 - a^2}) - \int \frac{x^2}{\sqrt{x^2 - a^2}}\, dx = x(\sqrt{x^2 - a^2}) - \int \frac{(x^2 - a^2) + a^2}{\sqrt{x^2 - a^2}}\, dx$$

$$= x(\sqrt{x^2 - a^2}) - \int \sqrt{x^2 - a^2}\, dx - \int \frac{a^2}{\sqrt{x^2 - a^2}}\, dx$$

$$= x(\sqrt{x^2 - a^2}) - I - a^2 \log\left|x + \sqrt{x^2 - a^2}\right| + c.$$

$$\therefore \quad 2I = x(\sqrt{x^2 - a^2}) - a^2 \log\left|x + \sqrt{x^2 - a^2}\right| + c$$

$$\Rightarrow I = \frac{x}{2}(\sqrt{x^2 - a^2}) - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C \quad \left[\text{taking } \frac{c}{2} = C\right].$$

Hence, $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}(\sqrt{x^2 - a^2}) - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C.$

(iii) Integrating by parts, taking 1 as the second function, we get

$$I = \int \sqrt{x^2 + a^2}\,dx = \int (\sqrt{x^2 + a^2} \cdot 1)\,dx$$

$$= (\sqrt{x^2 + a^2})x - \int \frac{1}{2}(x^2 + a^2)^{-1/2}(2x) \cdot x\,dx$$

$$= x(\sqrt{x^2 + a^2}) - \int \frac{x^2}{\sqrt{x^2 + a^2}}\,dx = x(\sqrt{x^2 + a^2}) - \int \frac{(x^2 + a^2) - a^2}{\sqrt{x^2 + a^2}}\,dx$$

$$= x(\sqrt{x^2 + a^2}) - \int \sqrt{x^2 + a^2}\,dx + a^2 \int \frac{dx}{\sqrt{x^2 + a^2}}$$

$$= x \cdot (\sqrt{x^2 + a^2}) - I + a^2 \log \left| x + \sqrt{x^2 + a^2} \right| + c.$$

$$\therefore \quad 2I = x(\sqrt{x^2 + a^2}) + a^2 \log \left| x + \sqrt{x^2 + a^2} \right| + c$$

or $\quad I = \frac{x}{2}(\sqrt{x^2 + a^2}) + \frac{a^2}{2} \log \left| x + \sqrt{x^2 + a^2} \right| + C \qquad$ [taking $\frac{c}{2} = C$].

Hence, $\quad \int \sqrt{x^2 + a^2}\,dx = \frac{x}{2}(\sqrt{x^2 + a^2}) + \frac{a^2}{2} \log \left| x + \sqrt{x^2 + a^2} \right| + C.$

## SOLVED EXAMPLES

EXAMPLE 1 *Evaluate:*

*(i)* $\int \sqrt{9 - x^2}\,dx \qquad$ *(ii)* $\int \sqrt{1 - 4x^2}\,dx \qquad$ *(iii)* $\int \sqrt{16 - 9x^2}\,dx$

SOLUTION We know that $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C.$

$\therefore$ (i) $\int \sqrt{9 - x^2}\,dx = \int \sqrt{3^2 - x^2}\,dx$

$$= \frac{x}{2}\sqrt{9 - x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3} + C.$$

(ii) $\sqrt{1 - 4x^2}\,dx = 2\int \sqrt{\left(\frac{1}{4} - x^2\right)}\,dx = 2\int \left\{ \sqrt{\left(\frac{1}{2}\right)^2 - x^2} \right\} dx$

$$= 2\left[ \frac{x}{2}\sqrt{\frac{1}{4} - x^2} + \frac{1}{8}\sin^{-1}\left( \frac{x}{(1/2)} \right) \right] + C$$

$$= \frac{x}{2}\sqrt{1 - 4x^2} + \frac{1}{4}\sin^{-1}(2x) + C.$$

(iii) $\int \sqrt{16 - 9x^2}\,dx = 3\int \left\{ \sqrt{\left( \frac{16}{9} - x^2 \right)} \right\} dx = 3\int \left\{ \sqrt{\left( \frac{4}{3} \right)^2 - x^2} \right\} dx$

$$= 3\left[ \frac{x}{2}\sqrt{\frac{16}{9} - x^2} + \frac{8}{9}\sin^{-1}\frac{x}{(4/3)} \right] + C$$

$$= \frac{x}{2}\sqrt{16 - 9x^2} + \frac{8}{3}\sin^{-1}\left( \frac{3x}{4} \right) + C.$$

EXAMPLE 2 *Evaluate:*

(i) $\int \sqrt{x^2-16}\,dx$ (ii) $\int \sqrt{4x^2-5}\,dx$ (iii) $\int \sqrt{17x^2-11}\,dx$

SOLUTION We know that $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C.$

$\therefore$ (i) $\int \sqrt{x^2-16}\,dx = \int \sqrt{x^2-4^2}\,dx$

$$= \frac{x}{2}\cdot\sqrt{x^2-4^2} - \frac{16}{2}\log\left|x+\sqrt{x^2-16}\right| + C$$

$$= \frac{x}{2}\cdot\sqrt{x^2-16} - 8\log\left|x+\sqrt{x^2-16}\right| + C.$$

(ii) $\int \sqrt{4x^2-5}\,dx = 2\int \sqrt{x^2-\frac{5}{4}}\,dx = 2\cdot\int \sqrt{x^2-\left(\frac{\sqrt{5}}{2}\right)^2}\,dx$

$$= 2\cdot\left[\frac{x}{2}\sqrt{x^2-\frac{5}{4}} - \frac{5}{8}\log\left|x+\sqrt{x^2-\frac{5}{4}}\right|\right] + C$$

$$= x\sqrt{x^2-\frac{5}{4}} - \frac{5}{4}\log\left|x+\sqrt{x^2-\frac{5}{4}}\right| + C.$$

(iii) $\int \sqrt{17x^2-11}\,dx = \sqrt{17}\cdot\int \sqrt{x^2-\frac{11}{17}}\,dx$

$$= \sqrt{17}\cdot\left\{\frac{x}{2}\sqrt{x^2-\frac{11}{17}} - \frac{11}{34}\log\left|x+\sqrt{x^2-\frac{11}{17}}\right|\right\} + C$$

$$= \frac{x}{2}\sqrt{17x^2-11} - \frac{11\sqrt{17}}{34}\log\left|x+\sqrt{x^2-\frac{11}{17}}\right| + C.$$

EXAMPLE 3 *Evaluate* $\int \sqrt{16x^2+25}\,dx.$

SOLUTION We know that $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C.$

$\therefore$ $\int \sqrt{16x^2+25}\,dx = 4\int\left\{\sqrt{x^2+\frac{25}{16}}\right\}dx = 4\int\left\{\sqrt{x^2+\left(\frac{5}{4}\right)^2}\right\}dx$

$$= 4\cdot\left\{\frac{x}{2}\sqrt{x^2+\frac{25}{16}} + \frac{25}{32}\log\left|x+\sqrt{x^2+\frac{25}{16}}\right|\right\} + C$$

$$= \frac{x}{2}\cdot\sqrt{16x^2+25} + \frac{25}{8}\log\left|x+\sqrt{x^2+\frac{25}{16}}\right| + C.$$

EXAMPLE 4 *Evaluate* $\int \sqrt{\frac{16+(\log x)^2}{x}}\,dx.$ **[CBSE 2005]**

SOLUTION Putting $\log x = t$ and $\frac{1}{x}\,dx = dt$, we get

$I = \int \sqrt{16+t^2}\,dt = \int \sqrt{4^2+t^2}\,dt$

$$= \frac{t}{2}\sqrt{16+t^2} + \frac{16}{2}\log\left|t+\sqrt{16+t^2}\right| + C$$

$$= \frac{1}{2}\log x \cdot \sqrt{16+(\log x)^2} + 8\log\left|\log x + \sqrt{16+(\log x)^2}\right| + C.$$

EXAMPLE 5 *Evaluate* $\int x\sqrt{x^4+1}\,dx.$ **[CBSE 2003]**

SOLUTION Putting $x^2 = t$ and $x\,dx = \frac{1}{2}\,dt$, we get

$$I = \frac{1}{2}\int\sqrt{t^2+1}\,dt$$

$$= \frac{1}{2}\cdot\left[\frac{t}{2}\sqrt{t^2+1} + \frac{1}{2}\log\left|t+\sqrt{t^2+1}\right|\right] + C$$

$$= \frac{x^2}{4}\sqrt{x^4+1} + \frac{1}{4}\log\left|x^2+\sqrt{x^4+1}\right| + C.$$

EXAMPLE 6 *Evaluate* $\int e^x\sqrt{e^{2x}+4}\,dx.$

SOLUTION Putting $e^x = t$ and $e^x dx = dt$, we get

$$I = \int\sqrt{t^2+4}\,dx$$

$$= \frac{t}{2}\cdot\sqrt{t^2+4} + \frac{4}{2}\log\left|t+\sqrt{t^2+4}\right| + C$$

$$= \frac{1}{2}e^x\sqrt{e^{2x}+4} + 2\log\left|e^x+\sqrt{e^{2x}+4}\right| + C.$$

EXAMPLE 7 *Evaluate* $\int \cos x\sqrt{4-\sin^2 x}\,dx.$

SOLUTION Putting $\sin x = t$ and $\cos x\,dx = dt$, we get

$$I = \int\sqrt{4-t^2}\,dt$$

$$= \frac{t}{2}\sqrt{4-t^2} + \frac{4}{2}\sin^{-1}\frac{t}{2} + C$$

$$= \frac{1}{2}\sin x\sqrt{4-\sin^2 x} + 2\sin^{-1}\left(\frac{1}{2}\sin x\right) + C.$$

**Integrals of the form $\int\sqrt{(ax^2+bx+c)}\,dx$**

**METHOD** Express $(ax^2+bx+c)$ as $a[(x+\alpha)^2 \pm \beta^2]$ and obtain an integral which can be evaluated easily.

EXAMPLE 8 *Evaluate* $\int\sqrt{x^2+3x}\,dx.$

SOLUTION We have

$$(x^2+3x) = \left\{x^2+3x+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2\right\} = \left(x+\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2.$$

$$\therefore \quad I = \int \sqrt{x^2 + 3x}\ dx$$

$$= \int \sqrt{\left(x + \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2}\ dx = \int \sqrt{t^2 - \left(\frac{3}{2}\right)^2}\ dt, \text{ where } \left(x + \frac{3}{2}\right) = t$$

$$= \frac{1}{2} t \sqrt{t^2 - \frac{9}{4}} - \frac{9}{8} \log \left| t + \sqrt{t^2 - \frac{9}{4}} \right| + C$$

$$\left\{ \text{using } \int \sqrt{x^2 - a^2}\ dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left| x + \sqrt{x^2 - a^2} \right| + C \right\}$$

$$= \frac{1}{2}\left(x + \frac{3}{2}\right) \sqrt{x^2 + 3x} - \frac{9}{8} \log \left| \left(x + \frac{3}{2}\right) + \sqrt{x^2 + 3x} \right| + C.$$

EXAMPLE 9 *Evaluate* $\int \sqrt{2x^2 + 3x + 4}\ dx.$

SOLUTION We have

$$(2x^2 + 3x + 4) = 2\left(x^2 + \frac{3}{2}x + 2\right)$$

$$= 2 \cdot \left\{ \left(x^2 + \frac{3}{2}x + \frac{9}{16}\right) + \left(2 - \frac{9}{16}\right) \right\} = 2 \cdot \left\{ \left(x + \frac{3}{4}\right)^2 + \left(\frac{\sqrt{23}}{4}\right)^2 \right\}.$$

$$\therefore \quad \sqrt{2x^2 + 3x + 4} = \sqrt{2} \cdot \sqrt{\left(x + \frac{3}{4}\right)^2 + \left(\frac{\sqrt{23}}{4}\right)^2}$$

$$\Rightarrow \int \sqrt{2x^2 + 3x + 4}\, dx = \sqrt{2} \cdot \int \sqrt{\left(x + \frac{3}{4}\right)^2 + \left(\frac{\sqrt{23}}{4}\right)^2}\ dx$$

$$= \sqrt{2} \int \sqrt{t^2 + \left(\frac{\sqrt{23}}{4}\right)^2}\ dt, \text{ where } \left(x + \frac{3}{4}\right) = t$$

$$= \sqrt{2} \cdot \left\{ \frac{t}{2} \cdot \sqrt{t^2 + \frac{23}{16}} + \frac{23}{32} \log \left| t + \sqrt{t^2 + \frac{23}{16}} \right| \right\} + C$$

$$\left[ \because \int \sqrt{t^2 + a^2}\, dt = \frac{t}{2} \sqrt{t^2 + a^2} + \frac{a^2}{2} \log \left| t + \sqrt{t^2 + a^2} \right| + C \right]$$

$$= \frac{\sqrt{2}}{2}\left(x + \frac{3}{4}\right) \sqrt{\left(x + \frac{3}{4}\right)^2 + \frac{23}{16}}$$

$$+ \frac{23\sqrt{2}}{32} \log \left| \left(x + \frac{3}{4}\right) + \sqrt{x^2 + \frac{3}{2}x + 2} \right| + C$$

$$= \frac{(4x + 3)}{4\sqrt{2}} \cdot \sqrt{x^2 + \frac{3}{2}x + 2} + \frac{23\sqrt{2}}{32} \log \left| \frac{(4x + 3)}{4} + \frac{\sqrt{2x^2 + 3x + 4}}{\sqrt{2}} \right| + C$$

$$= \frac{(4x + 3)\sqrt{2x^2 + 3x + 4}}{8} + \frac{23\sqrt{2}}{32} \log \left| \frac{(4x + 3)}{4} + \frac{\sqrt{2x^2 + 3x + 4}}{\sqrt{2}} \right| + C$$

EXAMPLE 10 *Evaluate* $\int \sqrt{3-2x-2x^2}\,dx.$

SOLUTION We have

$$(3-2x-2x^2) = 2\cdot\left\{\frac{3}{2}-x-x^2\right\}$$

$$= 2\cdot\left[\frac{3}{2}-\left(x^2+x+\frac{1}{4}\right)+\frac{1}{4}\right]$$

$$= 2\cdot\left[\frac{7}{4}-\left(x+\frac{1}{2}\right)^2\right] = 2\cdot\left\{\left(\frac{\sqrt{7}}{2}\right)^2-\left(x+\frac{1}{2}\right)^2\right\}.$$

$$\therefore\ \sqrt{3-2x-2x^2} = \sqrt{2}\cdot\sqrt{\left(\frac{\sqrt{7}}{2}\right)^2-\left(x+\frac{1}{2}\right)^2}$$

$$\Rightarrow \int\sqrt{3-2x-2x^2}\,dx = \sqrt{2}\cdot\int\sqrt{\left(\frac{\sqrt{7}}{2}\right)^2-\left(x+\frac{1}{2}\right)^2}\,dx$$

$$= \sqrt{2}\cdot\int\sqrt{\left(\frac{\sqrt{7}}{2}\right)^2-t^2}\,dt,\ \text{where}\left(x+\frac{1}{2}\right)=t \text{ and } dx = dt$$

$$= \sqrt{2}\cdot\left\{\frac{t}{2}\cdot\sqrt{\frac{7}{4}-t^2}+\frac{7}{8}\sin^{-1}\frac{t}{(\sqrt{7}/2)}\right\}+C$$

$$\left[\because\ \int\sqrt{a^2-t^2}\,dt = \frac{t}{2}\sqrt{a^2-t^2}+\frac{a^2}{2}\cdot\sin^{-1}\frac{t}{a}+C\right]$$

$$= \sqrt{2}\cdot\left\{\frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{\frac{7}{4}-\left(x+\frac{1}{2}\right)^2}+\frac{7}{8}\sin^{-1}\frac{\left(x+\frac{1}{2}\right)}{(\sqrt{7}/2)}\right\}+C$$

$$= \sqrt{2}\left\{\frac{1}{4}(2x+1)\cdot\sqrt{\frac{3}{2}-x-x^2}+\frac{7}{8}\sin^{-1}\left(\frac{2x+1}{\sqrt{7}}\right)\right\}+C$$

$$= \frac{1}{4}(2x+1)\sqrt{3-2x-2x^2}+\frac{7}{4\sqrt{2}}\sin^{-1}\left(\frac{2x+1}{\sqrt{7}}\right)+C.$$

**Integrals of the form** $\int (px+q)\sqrt{(ax^2+bx+c)}\,dx$

**METHOD** Let $(px+q) = A\cdot\frac{d}{dx}(ax^2+bx+c)+B.$

Find $A$ and $B$.

Then, we get the integrand which is easily integrable.

EXAMPLE 11 *Evaluate* $\int (x+3)\sqrt{3-4x-x^2}\,dx.$ **[CBSE 2005C]**

SOLUTION Let $(x+3) = A\cdot\frac{d}{dx}(3-4x-x^2)+B \Leftrightarrow (x+3) = A(-4-2x)+B.$

Comparing the coefficients of like powers of $x$, we get

$$-2A=1 \text{ and } -4A+B=3 \Leftrightarrow A=-\frac{1}{2} \text{ and } B=1.$$

$\therefore \quad (x+3)=-\frac{1}{2}(-4-2x)+1$

$\Rightarrow \quad I=\int\left\{-\frac{1}{2}(-4-2x)+1\right\}\sqrt{3-4x-x^2}\,dx$

$=-\frac{1}{2}\int(-4-2x)\sqrt{3-4x-x^2}\,dx+\int\sqrt{3-4x-x^2}\,dx$

$=-\frac{1}{2}\int\sqrt{t}\,dt+\int\sqrt{7-(4x+x^2+4)}\,dx$, where $3-4x-x^2=t$

$=\left(-\frac{1}{2}\times t^{3/2}\times\frac{2}{3}\right)+\int\sqrt{(\sqrt{7})^2-(x+2)^2}\,dx$

$=-\frac{1}{3}(3-4x-x^2)^{3/2}+\frac{1}{2}(x+2)\sqrt{3-4x-x^2}+\frac{7}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{7}}\right)+C$

$$\left[\because \int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\cdot\sin^{-1}\frac{x}{a}+C\right].$$

EXAMPLE 12 *Evaluate* $\int(3x-2)\sqrt{x^2+x+1}\,dx.$

SOLUTION Let $(3x-2)=A\cdot\frac{d}{dx}(x^2+x+1)+B.$

Then, $(3x-2)=A(2x+1)+B.$

Comparing the coefficients of like powers of $x$, we get

$$2A=3 \text{ and } A+B=-2. \text{ So, } A=\frac{3}{2} \text{ and } B=\frac{-7}{2}.$$

$\therefore \quad (3x-2)=\frac{3}{2}(2x+1)-\frac{7}{2}.$

So, $\int(3x-2)\sqrt{x^2+x+1}\,dx$

$=\int\left\{\frac{3}{2}(2x+1)-\frac{7}{2}\right\}\sqrt{x^2+x+1}\,dx$

$=\frac{3}{2}\int(2x+1)\sqrt{x^2+x+1}\,dx-\frac{7}{2}\int\sqrt{x^2+x+1}\,dx$

$=\frac{3}{2}\int\sqrt{t}\,dt-\frac{7}{2}\int\sqrt{\left(x^2+x+\frac{1}{4}\right)+\frac{3}{4}}\,dx,$

where $x^2+x+1=t$ in the 1st integral

$=t^{3/2}-\frac{7}{2}\int\sqrt{\left\{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2\right\}}\,dx$

$=(x^2+x+1)^{3/2}-\frac{7}{2}\cdot\int\sqrt{u^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,du$, where $u=\left(x+\frac{1}{2}\right)$

$$= (x^2 + x + 1)^{3/2} - \frac{7}{2}\left\{\frac{u}{2}\cdot\sqrt{u^2 + \frac{3}{4}} + \frac{3}{8}\log\left|u + \sqrt{u^2 + \frac{3}{4}}\right|\right\} + C$$

$$\left\{\because \int \sqrt{u^2 + a^2}\,du = \frac{u}{2}\sqrt{u^2 + a^2} + \frac{a^2}{2}\log\left|u + \sqrt{u^2 + a^2}\right| + C\right\}$$

$$= (x^2 + x + 1)^{3/2} - \frac{7u}{8}\sqrt{4u^2 + 3} - \frac{21}{16}\log\left|u + \sqrt{u^2 + \frac{3}{4}}\right| + C$$

$$= (x^2 + x + 1)^{3/2} - \frac{7}{8}\left(x + \frac{1}{2}\right)\sqrt{4\left(x + \frac{1}{2}\right)^2 + 3}$$

$$- \frac{21}{16}\log\left|\left(x + \frac{1}{2}\right) + \sqrt{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}}\right| + C$$

$$= (x^2 + x + 1)^{3/2} - \frac{7(2x + 1)}{8}\sqrt{x^2 + x + 1}$$

$$- \frac{21}{16}\log\left|\frac{(2x + 1)}{2} + \sqrt{x^2 + x + 1}\right| + C.$$

EXAMPLE 13 *Evaluate* $\int x\sqrt{x + x^2}\,dx.$ **[CBSE 2005C]**

SOLUTION Let $x = A \cdot \frac{d}{dx}(x + x^2) + B$. Then,

$$x = A(1 + 2x) + B \qquad \ldots \text{(i)}$$

Comparing the coefficients of various powers of $x$, we get

$$(2A = 1 \text{ and } A + B = 0) \Rightarrow \left(A = \frac{1}{2} \text{ and } B = \frac{-1}{2}\right).$$

$$\therefore \quad x = \frac{1}{2}(1 + 2x) - \frac{1}{2}$$

$$\Rightarrow \int x\sqrt{x + x^2}\,dx$$

$$= \int \left\{\frac{1}{2}(1 + 2x) - \frac{1}{2}\right\}\sqrt{x + x^2}\,dx$$

$$= \frac{1}{2}\int (1 + 2x)\sqrt{x + x^2}\,dx - \frac{1}{2}\int \sqrt{x + x^2}\,dx$$

$$= \frac{1}{2}\int \sqrt{t}\,dt - \frac{1}{2}\int \sqrt{\left\{\left(x^2 + x + \frac{1}{4}\right) - \frac{1}{4}\right\}}\,dx,$$

where $(x + x^2) = t$ in the first integral

$$= \frac{1}{2}\cdot\frac{t^{3/2}}{(3/2)} - \frac{1}{2}\int \sqrt{\left\{\left(x + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right\}}\,dx$$

$$= \frac{1}{3}(x + x^2)^{3/2} - \frac{1}{2}\cdot\int \sqrt{u^2 - \left(\frac{1}{2}\right)^2}\,du, \text{ where } \left(x + \frac{1}{2}\right) = u$$

$$= \frac{1}{3}(x+x^2)^{3/2} - \frac{1}{2}\left\{\frac{u}{2}\cdot\sqrt{u^2 - \frac{1}{4}} - \frac{1}{8}\log\left|u + \sqrt{u^2 - \frac{1}{4}}\right|\right\} + C$$

$$\left\{\because \int\sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C\right\}$$

$$= \frac{1}{3}(x+x^2)^{3/2} - \frac{1}{4}\left(x + \frac{1}{2}\right)\sqrt{\left(x + \frac{1}{2}\right)^2 - \frac{1}{4}}$$

$$+ \frac{1}{16}\log\left|\left(x + \frac{1}{2}\right)\sqrt{\left(x + \frac{1}{2}\right)^2 - \frac{1}{4}}\right| + C$$

$$= \frac{1}{3}(x+x^2)^{3/2} - \frac{1}{8}(2x+1)\sqrt{x+x^2} + \frac{1}{16}\log\left|\frac{(2x+1)}{2}\cdot\sqrt{x+x^2}\right| + C.$$

## EXERCISE 14C

*Evaluate the following integrals:*

1. $\int\sqrt{4-x^2}\,dx$ **[CBSE 2003C, '06]**
2. $\int\sqrt{4-9x^2}\,dx$
3. $\int\sqrt{x^2-2}\,dx$
4. $\int\sqrt{2x^2-3}\,dx$
5. $\int\sqrt{x^2+5}\,dx$
6. $\int\sqrt{4x^2+9}\,dx$
7. $\int\sqrt{3x^2+4}\,dx$
8. $\int\cos x\sqrt{9-\sin^2 x}\,dx$
9. $\int\sqrt{x^2-4x+2}\,dx$
10. $\sqrt{x^2+6x-4}\,dx$
11. $\int\sqrt{2x-x^2}\,dx$
12. $\int\sqrt{1-4x-x^2}\,dx$
13. $\int\sqrt{2ax-x^2}\,dx$
14. $\int\sqrt{2x^2+3x+4}\,dx$
15. $\int\sqrt{x^2+x}\,dx$
16. $\int\sqrt{x^2+x+1}\,dx$
17. $\int(2x-5)\sqrt{x^2-4x+3}\,dx$ **[CBSE 2005C]**
18. $\int(x+2)\sqrt{x^2+x+1}\,dx$
19. $\int(x-5)\sqrt{x^2+x}\,dx$
20. $\int(4x+1)\sqrt{x^2-x-2}\,dx$
21. $\int(x+1)\sqrt{2x^2+3}\,dx$
22. $\int x\sqrt{1+x-x^2}\,dx$
23. $\int(2x-5)\sqrt{2+3x-x^2}\,dx$
24. $\int(6x+5)\sqrt{6+x-2x^2}\,dx$
25. $\int(x+1)\sqrt{1-x-x^2}\,dx$
26. $\int(x-3)\sqrt{x^2+3x-18}\,dx$ **[CBSE 2014]**

### ANSWERS (EXERCISE 14C)

**1.** $\frac{x}{2}\cdot\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2} + C$

**2.** $\frac{x}{2}\cdot\sqrt{4-9x^2} + 2\sin^{-1}\left(\frac{3x}{2}\right) + C$

**3.** $\frac{1}{2}x\sqrt{x^2-2}-\log\left|x+\sqrt{x^2-2}\right|+C$

**4.** $\frac{x}{2}\cdot\sqrt{2x^2-3}-\frac{3}{2\sqrt{2}}\log\left|\sqrt{2}x+\sqrt{2x^2-3}\right|+C$

**5.** $\frac{x}{2}\cdot\sqrt{x^2+5}+\frac{5}{2}\log\left|x+\sqrt{x^2+5}\right|+C$

**6.** $\frac{x}{2}\cdot\sqrt{4x^2+9}+\frac{9}{4}\log\left|2x+\sqrt{4x^2+9}\right|+C$

**7.** $\frac{x}{2}\cdot\sqrt{3x^2+4}+\frac{2}{\sqrt{3}}\log\left|\sqrt{3}x+\sqrt{3x^2+4}\right|+C$

**8.** $\frac{\sin x}{2}\sqrt{9-\sin^2 x}+\frac{9}{2}\sin^{-1}\left(\frac{\sin x}{3}\right)+C$

**9.** $\frac{(x-2)}{2}\cdot\sqrt{x^2-4x+2}-\log\left|(x-2)+\sqrt{x^2-4x+2}\right|+C$

**10.** $\frac{1}{2}(x+3)\sqrt{x^2+6x-4}-\frac{13}{2}\log\left|(x+3)+\sqrt{x^2+6x-4}\right|+C$

**11.** $\frac{1}{2}(x-1)\sqrt{2x-x^2}+\frac{1}{2}\sin^{-1}(x-1)+C$

**12.** $\frac{1}{2}(x+2)\sqrt{1-4x-x^2}+\frac{5}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right)+C$

**13.** $\frac{1}{2}(x-a)\sqrt{2ax-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right)+C$

**14.** $\frac{1}{8}(4x+3)\sqrt{2x^2+3x+4}+\frac{23\sqrt{2}}{32}\log\left|\left(x+\frac{3}{4}\right)+\frac{1}{\sqrt{2}}\cdot\sqrt{2x^2+3x+4}\right|+C$

**15.** $\frac{1}{4}(2x+1)\sqrt{x^2+x}-\frac{1}{8}\log\left|x+\frac{1}{2}+\sqrt{x^2+x}\right|+C$

**16.** $\frac{1}{4}(2x+1)\sqrt{x^2+x+1}+\frac{3}{8}\log\left|x+\frac{1}{2}+\sqrt{x^2+x+1}\right|+C$

**17.** $\frac{2}{3}(x^2-4x+3)^{3/2}-\frac{1}{2}(x-2)\sqrt{x^2-4x+3}+\frac{1}{2}\log\left|x-2+\sqrt{x^2-4x+3}\right|+C$

**18.** $\frac{1}{3}(x^2+x+1)^{3/2}+\frac{3}{8}(2x+1)\sqrt{x^2+x+1}+\frac{9}{16}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right|+C$

**19.** $\frac{1}{3}(x^2+x)^{3/2}-\frac{11}{8}(2x+1)\sqrt{x^2+x}+\frac{11}{16}\log\left|x+\frac{1}{2}+\sqrt{x^2+x}\right|+C$

**20.** $\frac{4}{3}(x^2-x-2)^{3/2}+\frac{3}{4}(2x-1)\sqrt{x^2-x-2}-\frac{27}{8}\log\left|\left(x-\frac{1}{2}\right)+\sqrt{x^2-x-2}\right|+C$

**21.** $\frac{1}{6}(2x^2+3)^{3/2}+\frac{x}{2}\sqrt{2x^2+3}+\frac{3\sqrt{2}}{4}\log\left|\sqrt{2}x+\sqrt{2x^2+3}\right|+C$

**22.** $-\frac{1}{3}(1+x-x^2)^{3/2}+\frac{1}{8}(2x-1)\sqrt{1+x-x^2}+\frac{5}{16}\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)+C$

**23.** $-\frac{2}{3}(2+3x-x^2)^{3/2}-\frac{1}{2}(2x-3)\sqrt{2+3x-x^2}-\frac{17}{4}\sin^{-1}\left(\frac{2x-3}{\sqrt{17}}\right)+C$

**24.** $-(6+x-2x^2)^{3/2}+\frac{13}{16}(4x-1)\sqrt{6+x-2x^2}+\frac{637}{32\sqrt{2}}\sin^{-1}\left(\frac{4x-1}{7}\right)+C$

**25.** $-\frac{1}{3}(1-x-x^2)^{3/2}+\frac{1}{8}(2x+1)\sqrt{1-x-x^2}+\frac{5}{16}\sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right)+C$

**26.** $\frac{1}{3}(x^2+3x-18)^{3/2}-\frac{9}{8}\{(2x+3)\sqrt{x^2+3x-18}\}$

$$-\frac{81}{2}\log\left|x+\frac{3}{2}+\sqrt{x^2+3x-18}\right|+C$$

## *HINTS TO SOME SELECTED QUESTIONS (EXERCISE 14C)*

2. $I=3\cdot\int\sqrt{\frac{4}{9}-x^2}\,dx.$

4. $I=\sqrt{2}\cdot\int\sqrt{x^2-\frac{3}{2}}\,dx.$

6. $I=2\cdot\int\sqrt{x^2+\frac{9}{4}}\,dx.$

7. $I=\sqrt{3}\cdot\int\sqrt{x^2+\frac{4}{3}}\,dx.$

8. Put $\sin x=t$ and $\cos x\,dx=dt$.

9. $I=\int\sqrt{(x-2)^2-(\sqrt{2})^2}\,dx.$

11. $I=\int\sqrt{-(x^2-2x+1)+1}\,dx=\int\sqrt{1-(x-1)^2}\,dx.$

12. $I=\int\sqrt{1-(x^2+4x+4)+4}\,dx=\int\sqrt{5-(x+2)^2}\,dx.$

13. $I=\int\sqrt{-(x^2-2ax+a^2)+a^2}\,dx=\int\sqrt{a^2-(x-a)^2}\,dx.$

14. $I=\sqrt{2}\cdot\int\sqrt{\left(x^2+\frac{3}{2}x+2\right)}\,dx=\sqrt{2}\int\sqrt{\left\{\left(x+\frac{3}{4}\right)^2+\frac{23}{16}\right\}}\,dx.$

17. Let $(2x-5)=A\cdot\frac{d}{dx}(x^2-4x+3)+B=A(2x-4)+B.$

26. Let $(x-3)=A\cdot\frac{d}{dx}(x^2+3x-18)+B.$ Then,

$(x-3)=A(2x+3)+B.$ This gives $A=\frac{1}{2}$ and $B=\frac{-9}{2}\cdot$

$$\therefore\quad I=\frac{1}{2}\int(2x+3)\sqrt{x^2+3x-18}\,dx-\frac{9}{2}\int\sqrt{x^2+3x-18}\,dx$$

$$=\frac{1}{2}\int\sqrt{t}\,dt-\frac{9}{2}\int\sqrt{\left(x+\frac{3}{2}\right)^2-\left(\frac{9}{2}\right)^2}\,dx.$$